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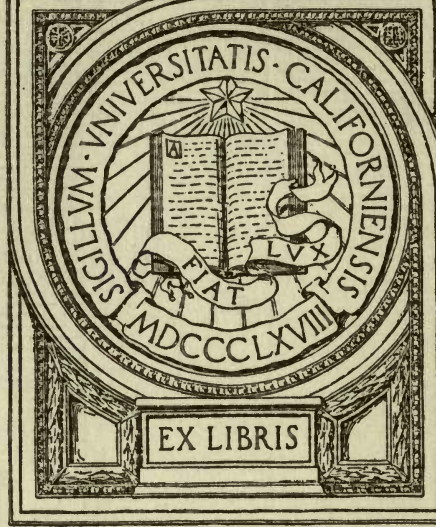

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A TREATISE

ON

ELECTRICITY AND MAGNETISM

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A TREATISE

ON

ELECTRICITY

AND

MAGNETISM

BY

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VOLUME I.

GENERAL PHENOMENA AND THEORY.

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PREFACE.

THIS work is based upon a course of Lectures delivered by one of us in the College de France in the last few years. It will comprise two parts: the first, which is principally theoretical, forms the present volume; the second, which will be more experimental in character, will be devoted to the examination of the various phenomena, the methods of measurement, and the principal applications.

This method of treatment seemed to us to present great advantages. The phenomena are, in fact, almost always very complicated, especially in the case of electricity; and to understand all the details which these phenomena involve often requires a more extensive knowledge than that furnished in the chapters with which they are more immediately connected. The explanation of the experiments will, therefore, be materially facilitated by a preliminary account of the general principles of the science.

After having stated and co-ordinated the facts which serve to establish the theory, we have investigated the mechanical consequences. This first volume forms, accordingly, a distinct work; and we might express the idea which has guided us by considering it as an *Essay on the Mechanical Theory of Electricity*, if such a title did not seem too ambitious.

We have endeavoured to bring into prominence the profound views introduced into science by Faraday, and

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so happily developed by Clerk Maxwell, on the consideration of the lines of force, and on the function of the medium in which electrical and magnetic actions are exerted. This conception brilliantly elucidates the relations between various phenomena, and has given rise to a totally unforeseen theory of light.

As our principal aim was to be of service to physicists, we have made all efforts to simplify the demonstrations without in any way sacrificing the strictness of the reasoning. Those parts which require a somewhat more advanced analysis, and which can be readily distinguished, may be omitted in a first reading; in most cases they are not indispensable for following the development of theory.

In the last few years the science of electricity has undergone a real transformation; we eagerly recognise our numerous obligations to the works of those physicists who have most contributed to this reform, and particularly to the memoirs of Sir W. Thomson, and the excellent treatise of Clerk Maxwell.

In this English edition we have introduced several corrections, and have given some new proofs of certain questions. We must express our obligations to Dr. Atkinson for the care and exactitude with which the translation has been made.

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A TREATISE

C. L. CORY.

ON

ELECTRICITY AND MAGNETISM

PART I.—STATICAL ELECTRICITY.

CHAPTER I.

INTRODUCTORY.

1. Most bodies, when rubbed, acquire, at any rate temporarily, the property of attracting light bodies. They are then said to be *electrified.* If the attracted body comes in contact with the electrified body, it is sooner or later repelled, and it is then itself found to be electrified. Electrical properties can accordingly be transmitted from one body to another by mere contact.

On the other hand, bodies which show no electrical properties are said to be in the natural state, or in the *neutral* state.

2. Conductors. — Insulators. — On certain bodies, such as glass, resin, silk, caoutchouc, etc., electricity remains localised for a shorter or longer time, at the place where it has been produced by friction, or by contact. Such bodies are said to be *bad conductors* of electricity.

On other bodies, on the contrary, such as the metals, electrical properties imparted to any one part, are almost instantaneously transmitted to all parts; these are said to be *conductors.* Most substances composing the earth belong to this latter class; air, and vapours, and generally all gases, belong to the former. Electricity can only be retained on a conductor by *insulating* it from the ground—that is, supporting it by a bad conductor, such as a rod of glass, or sealing-wax, or ebonite, or by silk threads. Hence we have the term *insulators*, applied to bad conductors.

The distinction between good and bad conductors does not really correspond to any essential difference of properties. Electricity moves upon all bodies with greater or less freedom. No bodies are known which are absolute insulators—that is to say, on which electrical properties can be retained for an indefinite time without alteration.

In like manner, notwithstanding the rapidity with which electricity is transmitted on the best conductors, there are none on which it is propagated instantaneously. Very material differences are found among them in this respect, and we can determine the special resistance which each one offers to the motion of electricity.

3. Two Electricities.—When any two bodies are rubbed together, a piece of glass and a piece of resin for instance, both become electrified, but with different characters; each of them repels an insulated light body which has been in contact with it, and which has shared its electricity; but the resin attracts the body which has been touched by the glass, and the glass the body which has been touched by the resin.

The condition of the glass differs, then, from that of the resin, which is expressed by saying that the electricity of the glass is of a *different kind* to that of the resin. Experiment shows, moreover, that any electrified body behaves either like the glass, or like the resin of the preceding experiment. It attracts, for instance, the body electrified by glass, and repels that which has been electrified by resin, or conversely. There are thus two kinds of electricity, and only two. This fundamental property may be formulated by saying that *two bodies charged with the same electricity repel each other, and two bodies charged with opposite electricities attract each other.*

4. Electrical Actions.—Electrical Masses.—The action exerted between two electrical bodies whose dimensions are small in comparison with their distance apart, is in the direction of the straight line joining them.

Coulomb found by direct experiment that *this force is inversely as the square of the distance.* It is also a function of the electrical condition of adjacent bodies, or of their electrification.

If, between two identical bodies of very small dimensions, and placed at unit distance, the electrical action is equal to unit force, the *quantity of electricity*, or the *electrical mass* of each of them, is equal to unity.

If, while the condition of one of the bodies remains unchanged, the distance being also unchanged, the action between them becomes 2, 3... times as great, the electrical mass of the other is said to have become 2, 3... times as great.

The electrical mass of a body, other things being equal, is (proportional to the force which it exerts upon an external body placed at a distance) considerable in reference to its dimensions; and the mutual action of two electrical masses is proportional to the product of their electrical masses.

5. When two bodies—for instance, a glass disc and one of metal, the latter being insulated—are kept in contact after being rubbed together, the whole system behaves in reference to an external body, electrified or not, just as if it were in the neutral state. The electrical properties developed by friction have not, however, disappeared; for if the two discs are separated, it may at once be shown that each of them is electrified. The actions of the two bodies in contact are accordingly equal and of opposite signs. Hence follows this double conclusion.

By their mutual friction two bodies acquire quantities of electricity, which are equal and of different kinds.

The law according to which the action varies with the distance is the same for the two electricities.

6. Electrical Signs.—We are thus led to consider electrical masses of different kinds, as quantities of the same nature and of opposite signs. When a closed surface contains electrical masses of different kinds, the action exerted upon an external mass, equal to unity, and at a great distance, as compared with the surface in question, is proportional to the difference of the electrical masses of each kind, and is attractive or repulsive according to the kind of electricity which predominates. Affixing to these masses the signs + and −, we may say that the resultant action is proportional to the algebraical sum of the electrical masses contained on the surface, and is repulsive or attractive according to the sign of this sum.

It is usual to apply the term *positive* to the electricity developed upon glass when rubbed with resin, and *negative* to the electricity acquired by the resin.

7. Electrical Force.—The action between two bodies of small dimensions, charged respectively with the masses m and m', and at the distance r from each other, is therefore

$$f = \frac{mm'}{r^2}.$$

This expression is positive if the two masses are of the same sign, and the force is then repulsive. In the contrary case it is attractive.

If a mass m is in the presence of any electrified bodies whatever, it may be considered that the total action which it experiences is the

resultant of all the actions which each of the elementary masses, considered separately, would exert upon it; and this whether the masses belong to separate bodies, or whether they form part of the charge of one and the same body.

For the sake of abbreviation, we shall apply the term, *electrical force at a point*, to the resultant of all the actions which would be exerted on unit mass of electricity placed at this point.

8. Distribution of Electricity.—Coulomb proved by direct experiment that when an electrified conducting sphere is put in contact with an identical sphere in the neutral state, each of them possesses a mass of electricity equal to half the original mass—that is to say, that each of them acting separately at the same distance upon an external electrified body, exerts half the action of that of the sphere in its original state. If the same sphere, instead of being neutral, is itself electrified before contact, the final charges are also equal; each of them is half the algebraical sum of the original masses, so that it is zero, and the bodies are in the neutral state, if the initial charges were equal, and of opposite signs.

This would also be the case with two identical conductors of any given shape, which were made to touch, provided that they were symmetrical at the point of contact.

If the condition of symmetry be not fulfilled, the charges are no longer equal; but their algebraic sum is always equal to that of the original mass. This is a general fact, and applies to any number of bodies, however they may be placed in relation to each other; and provided that none of the conductors are put, even for a moment, in contact with the earth, the algebraical sum of the electrical masses of the system remains the same.

9. Electricity of Contact.—Volta discovered this most important fact, that the contact of two different metals, originally in the neutral state, or more generally, of any two bodies at the same temperature, is sufficient to place them in two different electrical states, and to charge them respectively with equal quantities of electricity of opposite signs.

Friction is only a particular case of contact. The cause which produces the electricity seems then to be the same in both cases.

It follows from Volta's discovery, that two conducting spheres of the same radius would only have equal charges after contact, provided they were of the same material, and at the same temperature. But this is no exception to the fundamental proposition, that the algebraical sum of the charges is the same before and after contact.

10. Electrification by Influence.—Induction.—When a

body, originally in the neutral state, is placed near electrified bodies it becomes itself electrified; the phenomenon is known as *electrification by influence* or *induction*. If the body under electrical induction is insulated, its total electrical mass, from what we have seen, must remain zero. It will then be charged with two masses, equal, and of opposite signs, distributed according to a certain law. The phenomenon of induction always precedes the attraction of a neutral body by an electrified one, and the action which is exerted is simply that between the electrical masses. We may thus consider it as an experimental fact that there is never any direct action, except that of electrical masses on other electrical masses.

11. Electrical Equilibrium.—The essential characteristic of induction is that electricity is produced at every point of a conductor at which electrical force is exerted. Equilibrium can therefore only exist on a conductor, provided that the electrical force is zero at each of its points; the electricity which it possesses exerts, at each point of its surface, an action equal and of opposite sign to that of the external masses.

The necessary and sufficient condition for electrical equilibrium in a system of conductors, insulated or not, is then *that the electrical force be zero at any point whatever of each of them.*

12. Dielectrics.—Electrical force can, therefore, only exist in a state of equilibrium, on bad conductors, or insulators. For this reason Faraday gave the name of *dielectrics* to these bodies, to denote that they are bodies in which electrical forces may exist or be transmitted.

13. Localisation of Electricity on the Surface of Conductors.—The experiments of Cavendish and of Coulomb showed that in any electrical system in equilibrium, *conductors have electricity on their external surface only.* The surface of any closed cavity hollowed in a conductor, and not containing electrical masses, is destitute of electricity, and the electrical force is null throughout the above extent of the cavity. We shall find that this fundamental property is only compatible with the law of inverse squares.

14. Induction on a Closed Conductor.—This localisation of electricity on the surface of a conductor leads to several important consequences.

When a conductor is electrified *by induction*, each of the positive and negative layers, with which it is charged, forms a mass less than, or at most equal to, that of the influencing body, or *inductor*. When the influenced, or *induced*, conductor completely surrounds the inductor, the outer electrical layer is of the same kind as that of the

inductor, and the charge at each point is independent of the position of this latter. Nothing is altered, even if the inductor comes in contact with the internal surface; but then if it also is a conductor it only forms with the induced body a single conducting mass, and the internal surface retains no charge of electricity. There was, accordingly, in the inside of the induced body, an electrical layer equal and of opposite sign to that of the inductor, and hence on the external surface a layer equal and of the same sign.

The quantity of electricity induced by an electrified body on a conductor which completely surrounds it, is thus equal to the quantity of inducing electricity. This property also holds if the inducing body is a bad conductor, and more generally if the electrical masses are distributed in any manner whatever in the cavity of the conductor.

15. ADDITION OF CHARGES.—We have seen above that the electrical charge of a conductor may be divided into two. In like manner any electrical masses whatever may be added to a conductor. It is sufficient for this if the conductor has a cavity almost entirely closed through which electrified conductors may be introduced, and which, by contact, transmit the electricities with which they are charged to the outer surface.

16. We may then increase or diminish at pleasure, the algebraic sum of the electrical masses contained in the interior of a closed surface, provided we introduce, or give exit to, positive or negative masses. It is, however, important to remark that, if no mass traverses the surface in one direction or the other, whatever be the actions to which the enclosed body is submitted—friction, induction, contact, physical or chemical actions, it is impossible to modify the total quantity of electricity of the system. We can neither create nor destroy, on any body, a determinate quantity of electricity, without at the same time creating or destroying, on the same body, or on another, an equal quantity of electricity of the opposite sign.

17. HYPOTHESES RESPECTING THE NATURE OF ELECTRICITY.—Electricity, defined and measured as we have explained above, is a magnitude of a particular kind, perfectly definite from the mechanical point of view, affected with a sign like a quantity of motion, and the theory of electrical phenomena may be established from experimental laws without having recourse to any hypothesis. From the facility with which electricity is transmitted in conductors, it has often been compared to a fluid, just as formerly the effects of thermal conductivity were explained by the propagation of a special fluid. The character of duality, which electrical phenomena present, has been accounted for in two ways.

According to Franklin, a body in the natural state contains a normal quantity of the electrical fluid, and it becomes positively or negatively electrified, according as its charge of fluid is increased or diminished by the action of external bodies. The attractions and repulsions of bodies are explained by the mutual repulsion of the fluids, and by the attraction which they exert upon ponderable masses.

The hypothesis of two fluids, devised by Symmer, and adopted, at any rate provisionally, by Coulomb, assumes that there are two different fluids, that the molecules of the same fluid repel, and that different fluids attract; and, finally, that in a body in the natural state, there are equivalent quantities of the two fluids forming the neutral fluid. A body is electrified positively or negatively, according as it contains an excess of one or the other fluid. Attractions and repulsions are explained in like manner by the actions which they exert between the fluids and the ponderable matter.

It is the least defect of these hypotheses that they are superfluous. As, moreover, experiment indicates no limit to the electrification of a body, we are led to the conclusion that the normal charge of a body on Franklin's theory, or that the mass of neutral fluid in the theory of two fluids, is unlimited—a conclusion that is manifestly in contradiction with the notion of a material fluid.

A certain number of expressions used in the study of electricity have originated in the idea of fluids; there is no inconvenience in retaining them, if we are careful to define them by the mathematical and experimental properties to which they correspond, with the object, as Coulomb expressed it, "of presenting the results of calculation and of experiment with the fewest elements possible, and not of indicating the true causes of electricity." *

18. Electrical Density.—The idea of a fluid has thus led to that of electrical density. If the electricity occupies the whole extent of a body, in the case of a dielectric for instance, and that it is distributed uniformly, *electrical density* is the quantity of electricity, defined as above, which exists in the unit of volume. If the distribution is irregular, the density at a point is the ratio of the electrical charge of an element of volume at the point to that of the volume itself.

Conductors only possess electricity on the surface. If the distribution is uniform, the superficial density is the quantity of electricity which exists upon the unit of surface. In the case of any given

* *Histoire de l'Academie des Sciences pour 1788*, p. 673.

distribution, the superficial density at this point, is the ratio of the charge of an element of surface taken about this point to the extent of the element.

On the hypothesis of fluids, it must be admitted that the layer of electricity on the surface has a certain thickness, and that it penetrates to a certain depth, which may be extremely small, in the conductor, or in the dielectric which surrounds it. As the thickness of this layer cannot be determined by experiment, holding to the same order of considerations, we may either suppose the density to vary with constant thickness, or suppose the density constant with variable thickness; in this case, the expressions *electric density* and *electric thickness at a point* are equivalent.

It will be seen that, apart from any idea of fluid, the expressions of electric density in volume, or superficial density, have a purely mathematical or experimental meaning quite apart from any hypothesis.

CHAPTER II.

ON POTENTIAL.

19. We shall assume, in the first place, in conformity with experiment, that the action of two electrified bodies, of small dimensions, takes place along the straight line joining them, and only varies with the distance; that it satisfies, in short, the definition of what are called *central* forces; and, lastly, by definition (4), that the action is proportional to the product of the quantities of electricity which the bodies possess.

We shall assume, moreover, that the reciprocal action of two electrified bodies of finite dimensions is the resultant of the actions which would be exerted, according to the same function of the distances, between the elementary masses which make up the charge.

20. Electrical Field.—The term *electrical field* is applied to the entire extent of the space throughout which the action of any given electrical system is exerted. An electrical field is generally unlimited; it may be bounded in the case, for instance, in which all the acting masses are inside an entirely closed conductor. For masses whose magnitude and position are defined, the electrical force at each point of the field is merely a function of the co-ordinates of the point.

The force is zero in all conductors in a state of equilibrium; the electrical field does not comprise the volumes of conductors—it is formed of intermediate spaces occupied by an insulating medium or *dielectric.*

21. Lines of Force.—A *line of force* in an electrical field is a line tangential at each point to the direction of the force. Such a line is obviously continuous, so long as it does not encounter acting masses.

22. Definition of Potential.—Consider a system in equilibrium, and suppose that all the acting masses being fixed in their several positions, we move unit mass of positive electricity from

A to B. The work of the electrical forces which corresponds to this displacement, is independent of the path following in passing from A to B. This is a necessary consequence of the hypothesis (19) that the forces are central; for if it were otherwise, it is obvious that, by moving an electrical mass on suitable paths between the points A and B, we might produce an indefinite quantity of work, without a corresponding expenditure. The work in question only depends, then, on the co-ordinates of the points A and B; it is equal to the difference of the values V_A and V_B which the same function V has at these two points, and representing this work by W_A^B, we may write—

$$W_A^B = V_A - V_B. \tag{1}$$

The function V plays a paramount part in the study of electrical phenomena; it has been called *potential.* As this function is only defined by an integral, its value is only determined to within a constant, and the variations are measured by the electrical work.

From equation (1), *the excess of the potential at a point* A *over the potential at* B, *is equal to the work done by the electrical actions on unit mass in passing from* A *to* B; *or conversely, it is the work which must be expended against electrical force to move this mass from* B *to* A.

If unit mass moves along a line of force, the work for an infinitely small displacement *ds* is F*ds*, and the total work from A to B is expressed by

$$W_A^B = \int_A^B F ds. \tag{2}$$

23. Equipotential Surfaces.—Electromotive Force.—A *level surface*, or *equipotential surface*, is a surface perpendicular at every point to the direction of the force; that is to say, a surface which is perpendicular to all the lines of force which it meets. In the case of central forces, a surface satisfying this condition can always be drawn through any given point. If an electrical mass moves along such a surface, the elementary work is constantly zero, for the force is always perpendicular to the displacement. The potential has the same value for all points of the same electrical level.

Let us consider two equipotential surfaces, S_1 and S_2, whose potentials are respectively V_1 and V_2. The work corresponding to the displacement of unit mass from a point of the former to a point of the latter, has the value $V_1 - V_2$; it is independent of the path traversed, and even of the position of the point of departure from, and of the point of arrival at, the two surfaces.

The work done by a mass m of electricity in passing from the equipotential surface S_1 to the surface S_2, is $m(V_1 - V_2)$. Electrical work, like that of gravity on a falling body, appears as a product of two factors, one m, which corresponds to the weight of the body, and the other, $V_1 - V_2$, to the height of the fall.

When a mass of positive electricity is left to itself, it tends to move along a line of force towards the points where the potential is lower; negative electricity would move towards high potentials.

If the electrical masses are distributed on dielectrics, they can only be displaced by carrying along with them the dielectric itself. Conductors, on the contrary, are characterised by the property of allowing a free passage to electrical masses, which go to the surface, and distribute themselves there so as to produce equilibrium.

In all cases, the difference of potential $V_1 - V_2$ may be considered as producing the motion of electrical masses; it is often called the *electromotive force.*

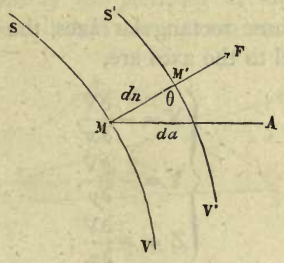

Fig. 1.

24. Expression of Force as a Function of Potential.—Consider two infinitely near equipotential surfaces S and S′, whose potentials are V and V′ (Fig. 1). At the point M of the former surface the force is F; if dn is the distance of the two surfaces measured along the perpendicular, the work done by this force on unit mass in going from M to M′ is equal to Fdn. We have then the equation—

$$Fdn = V - V' = -dV,$$

which gives—

$$F = -\frac{dV}{dn}. \tag{3}$$

Thus, *the force at a point is equal, and of opposite sign, to the differential of the potential, in reference to the perpendicular to the equipotential surface which passes through this point.*

The components of the force with reference to any given axis possess the same property. For let a line MA be drawn through the point M, making an angle θ with the perpendicular, and let *da* denote the portion of this line between the two surfaces S and S'. The component F_a of the force parallel to the straight line is thus expressed:—

$$F_a = F\cos\theta = -\frac{dV}{dn}\cos\theta.$$

The figure gives moreover,

$$dn = da\cos\theta;$$

from which we get,

$$F_a = -\frac{dV}{dn}\cdot\frac{dn}{da} = -\frac{\partial V}{\partial a}.$$

Thus, *the component of the force in any given direction is equal, and of opposite sign, to the partial differential of the potential along this direction.*

If we consider three rectangular axes, the components X, Y and Z of the force parallel to the axes are,

$$(4) \qquad \begin{cases} X = -\dfrac{\partial V}{\partial x}, \\ Y = -\dfrac{\partial V}{\partial y}, \\ Z = -\dfrac{\partial V}{\partial z}; \end{cases}$$

from which we have,

$$(5) \qquad F^2 = X^2 + Y^2 + Z^2 = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2.$$

25. EQUILIBRIUM OF CONDUCTORS.—In the interior of a conductor in equilibrium the force is zero (11). Hence for the whole surface of the conductors we have,

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} = 0, \qquad \frac{\partial V}{\partial z} = 0,$$

and consequently,

$$V = \text{constant}.$$

It follows from this, that the whole volume *of a conductor in equilibrium is at the same potential;* this is what may be called a *level volume.* Its surface being then a level surface, the

force is perpendicular at all points; hence, *the lines of force proceed perpendicularly from the conductors, or terminate there perpendicularly.*

26. NUMERICAL VALUE OF POTENTIAL.—In all these phenomena, equipotential surfaces are only apparent as differences, and not as the absolute values of the corresponding potentials. We may accordingly add to these potentials any given constant.

In the expression

$$W_1^2 = V_1 - V_2,$$

which represents the work corresponding to the displacement of a unit of electricity from an equipotential surface V_1, to an equipotential surface V_2, let us suppose that the latter is the earth, and that we agree to take its potential as equal to zero, we shall have

$$W_1^0 = V_1.$$

The numerical value of the potential at any given point, is the number of units of work which corresponds to the displacement of a

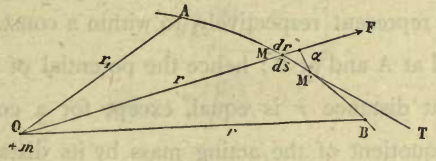

Fig. 2.

unit of positive electricity from this point, to the earth, by any path whatever. The sign of the potential is that of the work of the electrical forces in this displacement.

In other words, the potential at a point, is the work which must be done to bring unit mass of electricity from the earth, or from a body in connection with the earth, to this point.

27. POTENTIAL IN THE CASE OF THE LAW OF THE SQUARE OF THE DISTANCES.—We have hitherto left undetermined the law, according to which the action of electrical masses varies with the distance. We shall assume in future that this law is the *inverse of the square of the distance*, conformably to the experiments of Coulomb. In this case, the potential is expressed simply as a function of the masses and of the distances.

Let us suppose, in the first place, that the electrical system is reduced to a mass $+m$ placed at a point O. If a mass equal to unity placed at a point M (Fig. 2) at a distance r from the former, is

moved through MM′ or ds, along any curve whose tangent MT makes the angle α with the direction of the force, the corresponding electrical work is

$$dW = F ds \cos \alpha = F dr ;$$

since the force is expressed by

$$F = \frac{m}{r^2},$$

we have

$$dW = \frac{m}{r^2} dr = -m d\frac{1}{r} = -d\frac{m}{r}.$$

If r_1 and r_2 represent the distances OA and OB, the work of displacing unit mass from A to B is

$$W_A^B = \frac{m}{r_1} - \frac{m}{r_2}.$$

Comparing this equation with equation (1) we see that the two terms $\frac{m}{r_1}$ and $\frac{m}{r_2}$ represent respectively, to within a constant, the value of the potential at A and at B; hence the potential of a single mass m at a point at distance r is equal, except for a constant, to $\frac{m}{r}$, that is, to the quotient of the acting mass by its distance from the point in question.

Let us now suppose that there are several acting masses m, m', m'', ..., the total work of the displacement of unit of electricity is equal to the algebraical sum of the partial works corresponding to each of the masses; denoting then by $\Sigma\frac{m}{r_1}$ the sum of the quotients of the different masses by their distances from the point of departure A, and by $\Sigma\frac{m}{r_2}$ the analogous sum for the point of arrival B,

$$W_A^B = \Sigma\frac{m}{r_1} - \Sigma\frac{m}{r_2} = V_A - V_B.$$

If the point B is in connection with the earth, the potential V_B is zero. On the other hand, the expression $\Sigma\frac{m}{r_2}$ becomes zero, if the point B is at a great distance from the masses in question, whether in the air or on the ground; and since the earth, like any conductor

in equilibrium, has everywhere the same potential, this expression, which implicitly contains masses relative to induced electricity, is also zero on the ground. The value of V_A is then reduced to $\Sigma\frac{m}{r}$.

In general, then, to express the potential V at any point of the field, we have

$$V = \Sigma\frac{m}{r}. \tag{6}$$

Thus, *the potential at a point is equal to the sum of the quotients obtained by dividing each of the acting masses by its distance from the given point.*

28. On the Flow of Force.—Let A (Fig. 3) be an equipotential surface, and dA an element of this surface at the point P. Through the contour of this element let lines of force be drawn

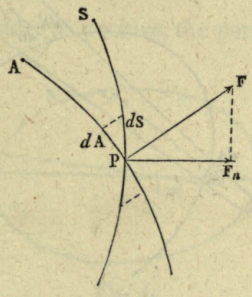

Fig. 3.

which cut all the successive equipotential surfaces at right angles. An orthogonal canal thus bounded by lines of force is called a *tube of force.*

The elementary tube of force, bounded by the contour of the element dA, cuts from any surface S passing through the point P an element of surface dS.

In P the electrical force F is perpendicular to dA; let F_n be the component of this force parallel to the perpendicular to the surface S; the angle of the two forces being equal to that of the two elements of surface, we have

$$\frac{F_n}{F} = \frac{dA}{dS},$$

whence

$$F\,dA = F_n\,dS.$$

If we suppose that a liquid, in the permanent state, traverses the element dA at right angles, with the velocity F, the product $F dA$ represents the volume of liquid which flows through the element dA n unit time, or, more briefly, the flow of liquid corresponding to this element. This flow is also expressed by the product $F_n dS$, obtained by multiplying any section of the tube by the component of the velocity along the perpendicular to this section.

By analogy, we shall apply the term *quantity of force, or flow of force corresponding to an element of surface, to the product* $F_n dS$ *of the surface of the element, into the component perpendicular to the force at this point.* The flow of force for unit surface is equal numerically to the perpendicular component of the force.

The properties which we are about to examine will completely justify this analogy.

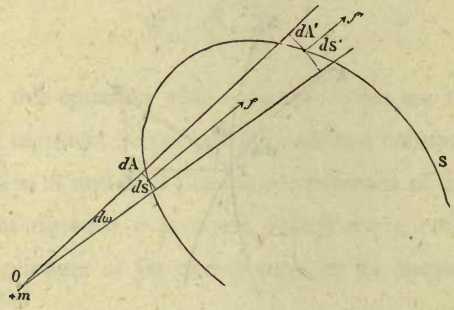

Fig. 4.

The idea of lines of force is due to Faraday, and this eminent physicist showed all the advantages which may be derived from it in the study of electrical phenomena. What we have designated a *quantity* or *flow of force,* Faraday called *number of lines of force.* It seemed useful to adopt another designation, in the first place for simplicity of expression, and secondly because the word *flow* seems to correspond better with the character of continuity which we want to determine.

29. GREEN'S THEOREM.—Let us consider in the dielectric an entirely closed convex surface S (Fig. 4), and let $+m$ be an electrical mass situate at the point O outside this surface An infinitely slender cone of aperture $d\omega$, having its apex at O, cuts the elements dS and dS' in this surface. Let f and f' be the values of the electrical force at dS and dS', dA and dA' the corresponding perpendicular

sections of the cone, and r and r' their distances from the point O. We have then

$$fr^2 = f'r'^2 = m,$$

$$\frac{d\mathrm{A}}{r^2} = \frac{d\mathrm{A}'}{r'^2} = d\omega,$$

and multiplying these equations by each other, we get

$$fd\mathrm{A} = f'd\mathrm{A}' = md\omega;$$

but in virtue of the theorem of the flow of force we have also

$$fd\mathrm{A} = f_n d\mathrm{S}, \quad f'd\mathrm{A}' = f'_n d\mathrm{S}',$$

and, therefore,

$$f_n d\mathrm{S} = f'_n d\mathrm{S}' = md\omega.$$

If we agree to consider as positive, the perpendicular components

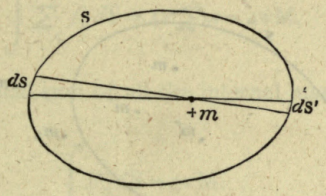

Fig. 5.

directed towards the exterior of the surface, and as negative those directed towards the interior, f_n and f'_n are of opposite signs, which gives

$$f_n d\mathrm{S} + f'_n d\mathrm{S}' = 0.$$

If the surface, while still continuous, had concave portions, and if the cone in question $d\omega$ cut it in more than two points, it would meet it an even number of times; the product $f_n d\mathrm{S}$ would have the same numerical value for each of the intercepted elements, but these products would have to be taken alternately of opposite signs, and the algebraical sum would still be zero. We have, then, for any closed surface external to the acting mass m, the equation

$$\int_{\mathrm{C}} f_n d\mathrm{S} = 0,$$

that is to say, that *the total flow of force which starts from the surface is equal to zero.*

If the acting mass m is within the closed surface S (Fig. 5), the elements dS and dS′, cut by a cone of aperture $d\omega$, starting from the mass m, always gives the ratio

$$f_n d\mathrm{S} = f'_n d\mathrm{S}' = m d\omega.$$

But in the present case, the perpendicular components f_n and f'_n are of the same sign. We have thus for the whole surface,

$$\int f_n d\mathrm{S} = m \int d\omega = 4\pi m.$$

The flow of force which proceeds from a surface S, *enclosing an acting mass, is thereby equal to* $4\pi m$. In other words, we may say that *the total flow of force which issues from a mass m, in all directions, is equal to* $4\pi m$.

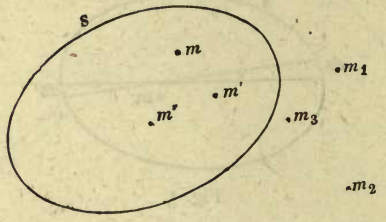

Fig. 6.

It is clear that if each sheet of the cone meets the surface more than once, it meets it an uneven number of times, for which the values of $f_n d$S should be taken alternately of opposite signs, and the final result is still the same.

30. Let us now suppose that there are masses m, m', m'', ... comprised within the surface S (Fig. 6), and other masses m_1, m_2, m_3, ... on the outside.

At each point of the surface, the perpendicular component F_n of the resultant force F, is equal to the algebraical sum of the perpendicular components of the forces proceeding from all the acting masses, both internal and external.

Calling $\sum_e f_n$ the sum, at a point, of the perpendicular components which arise from the external masses, and $\sum_i f_n$, the sum

of the components relative to the internal masses, the total flow of force which proceeds from the surface S is expressed by

$$\int F_n dS = \int \Sigma_e f_n dS + \int \Sigma_i f_n dS.$$

Since the flow of force is null for each of the external masses, we have

$$\int \Sigma_e f_n dS = 0.$$

For each internal mass, on the contrary, we have

$$\int f_n dS = 4\pi m,$$

and consequently

$$\int \Sigma_i f_n dS = 4\pi \Sigma m = 4\pi M,$$

M being the algebraical sum of all the internal masses.

We have, finally,

$$\int F_n dS = 4\pi M. \tag{7}$$

Hence, *for any closed surface, drawn in any manner in an electrical field, the total flow of force, which proceeds from the surface—that is, the excess of the flow of forces which emerge, over the forces which enter, is equal to the quantity of electricity, comprised within the surface, multiplied by* 4π.

If i be the angle which the direction of the force F at any point of the surface makes with the perpendicular, the perpendicular component is expressed by

$$F_n = F \cos i.$$

If a be the perpendicular to the equipotential surface which passes through the point in question, n the perpendicular to the surface S measured towards the exterior, we have, further,

$$F = -\frac{dV}{da} \quad \text{and} \quad F_n = -\frac{\partial V}{\partial n};$$

so that the preceding theorem may be expressed analytically by either of the equations

$$(8) \qquad -\int \frac{dV}{da} \cos i \, dS = 4\pi M,$$

$$-\int \frac{\partial V}{\partial n} dS = 4\pi M,$$

which are due to Green.

31. EQUATIONS OF LAPLACE AND OF POISSON.—Let X, Y and Z be the components of the force at a point P, whose co-ordinates are x, y, z, and let us consider the element of volume dx, dy, dz.

If the medium contains the acting masses distributed in a continuous manner, and if dM is the total mass contained in the element, then, denoting the density by ρ, we have

$$dM = \rho dxdydz.$$

The flow of force which enters by the surface $dydz$, passing through the point P, is

$$X dydz;$$

the flow of force which emerges from the opposite face is

$$\left(X + \frac{\partial X}{\partial x} dx\right) dydz.$$

The excess of the flow which emerges is equal to

$$\frac{\partial X}{\partial x} dxdydz = -\frac{\partial^2 V}{\partial x^2} dxdydz.$$

Repeating the same reasoning for the other co-ordinates, it will be seen that the total flow of force which proceeds from the element of volume is

$$-\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) dxdydz.$$

In virtue of the preceding theorem, this excess is equal to the product of 4π by the total mass of electricity comprised in the volume, which gives the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho.$$

Representing by ΔV the sum of the second partial differentials of the potential in reference to the co-ordinates, we may write:

$$\Delta V = -4\pi\rho. \tag{9}$$

If the element of volume is not electrified, $\rho = 0$ and

$$\Delta V = 0. \tag{10}$$

Thus, *the sum at a point of the three second partial differentials of the potential in reference to three rectangular axes is equal, and of opposite sign, to the product of* 4π *by the density of the mass acting at this point.*

This sum is zero when there is no electricity near the point.

This theorem of the second differentials was first enunciated by Laplace in the form (10). The more general equation is due to Poisson.

32. If the equipotential surfaces are concentric spheres, the force F is inversely as the square of the distance r from the common centre, and we have

$$F = -\frac{\partial V}{\partial r} = \frac{A}{r^2}, \qquad \frac{\partial^2 V}{\partial r^2} = \frac{2A}{r^2} = \frac{2F}{r}.$$

Taking the z axis along the perpendicular to the surface, the two others will be in the tangent plane; if we measure the distance in the direction of the force, we get

$$\frac{\partial^2 V}{\partial z^2} = -\frac{2F}{r},$$

and therefore, by Laplace's theorem,

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial y^2} = \frac{F}{r}.$$

If the equipotential surface is of any given form, it is readily seen that the second differential of the potential along the tangent to a principal section is the same as for the osculating circle. The z axis being always perpendicular, and the two others along the tangents to the principal sections, whose radii of curvature are R_1 and R_2, we shall have

$$\frac{\partial^2 V}{\partial x^2} = \frac{F}{R_1}, \qquad \frac{\partial^2 V}{\partial y^2} = \frac{F}{R_2},$$

and, therefore,

$$\frac{\partial^2 V}{\partial z^2} = -F\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

33. DISTRIBUTION OF ELECTRICITY ON THE SURFACE OF CONDUCTORS.—*In any conductor in equilibrium, electricity is present on the surface only.*

For we have seen that the force is zero in a conductor in equilibrium, and that therefore the potential has a constant value for the whole surface of the conductors. All orders of differentials of the potential are zero at each point, and we have then

$$\Delta V = 0, \qquad \text{or} \qquad \rho = 0.$$

Hence, in the interior of a conductor in equilibrium, not only is there no electrical force, but there is no electricity at all. The distribution is exclusively on the surface.

34. GREEN'S FORMULA.—The formula relative to the flow of force gave, for a closed surface, the equation

$$-\int \frac{\partial V}{\partial n} dS = 4\pi M.$$

The mass M, inside the surface, is equal to the sum of the masses ρdv comprised in the different elements of volume; taking equation (9) into account, we have then,

$$(11) \qquad \int \frac{\partial V}{\partial n} dS = -\int 4\pi\rho dv = \int \Delta V dv.$$

This equation is a particular case of a more general formula due to Green.

Let U and V be two finite and continuous functions of x, y, and z. Let us also put

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2},$$

and consider the integral

$$\int U \Delta V dv = \iiint U \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right) dx dy dz,$$

extended to the volume enveloped by a closed surface S. This integral consists of three terms of the form

$$\iiint U \frac{\partial^2 V}{\partial x^2} dx dy dz = \iint dy dz \int U \frac{\partial^2 V}{\partial x^2} dx.$$

Integrating this expression by parts, we have

$$\int\int dydz \int U\frac{\partial^2 V}{\partial x^2}dx = \int\int U\frac{\partial V}{\partial x}dydz - \int\int\int \frac{\partial U}{\partial x}.\frac{\partial V}{\partial x}dxdydz.$$

The first term of the second member should be extended to the whole surface S, and the second to the volume bounded by this surface. Repeating this operation for the other co-ordinates, the sum of the integrals relative to the surface S will be

$$\int\int U\left(\frac{\partial V}{\partial x}dydz + \frac{\partial V}{\partial y}dzdx + \frac{\partial V}{\partial z}dxdy\right).$$

Now, if we consider V as a potential, which does not restrict the general character of the demonstration, the expression $-\frac{\partial V}{\partial x}dydz$, or $Xdydz$, represents the flow of force through the surface element $dydz$; that is to say, the projection of a surface element dS on a plane perpendicular to the axis x. It is the same for other terms, so that, except for the sign, the parenthesis represents the excess of force which traverses this element of surface. This parenthesis is thus equal to $-F_n d$S, or $\frac{\partial V}{\partial n}d$S, and we have, finally, *Green's formula:*

$$(12)\quad \int U\Delta V dv = \int U\frac{\partial V}{\partial n}d\mathrm{S} - \int\left(\frac{\partial U}{\partial x}.\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}.\frac{\partial V}{\partial z}\right)dv.$$

Making U = 1, we obtain the preceding equation (11).

When the functions U and V are identical, we have

$$(13)\quad \int V\Delta V dv = \int V\frac{\partial V}{\partial n}d\mathrm{S} - \int\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]dv.$$

If the function V represents the potential of an electrical system, the force F is determined by the equation,

$$F^2 = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2,$$

which gives

$$(14)\quad \int V\Delta V dv = \int V\frac{\partial V}{\partial n}d\mathrm{S} - \int F^2 dv,$$

or, from Poisson's theorem,

$$(15) \qquad 4\pi \int V\rho dv = -\int V\frac{\partial V}{\partial n} dS + \int F^2 dv.$$

35. TUBES OF FORCE.—We have seen that a tube of force is an orthogonal canal with equipotential surfaces, bounded laterally by lines of force.

Let us consider a tube of force terminated by any two surfaces S and S′ (Fig. 7), and let us apply equation (7), to the volume thus defined. The lateral surface does not enter into the integral, for the perpendicular component of the force at each point is zero; the integral is reduced therefore to the two terms furnished by the terminal surfaces.

Let us first suppose that the tube contains no electrical mass.

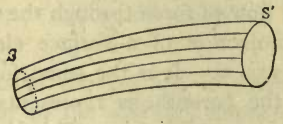

Fig. 7.

The integral reduced to the two terms of the bases must be zero, which gives

$$\int F_n dS + \int F'_n dS' = 0,$$

or, in absolute values,

$$\int F_n dS = \int F'_n dS';$$

that is to say that *the flow of force is then the same at both ends of the tube.*

It thus appears that the flow of force is the same across any section of the tube. This flow is kept up like the supply of a moving liquid, whose velocity at each point is equal and parallel to the direction of the force.

If the section of this tube is infinitely narrow, and if the terminal surfaces are perpendicular to the force, the equation reduces to

$$F dS = F' dS';$$

from which it follows that *the force at each point of the tube is inversely as the section.*

36. COULOMB'S THEOREM.—Let us consider any elementary surface dS, on a conductor in equilibrium, and let us draw the corresponding tube on the outside until it meets an infinitely near equipotential surface S_1. Let us prolong the tube inside the conductor in any manner, terminate it by any given surface S_2, and let us apply the theorem to the volume bounded by the surfaces S_1 and S_2, and the lateral surface of the tube. The force is zero on the whole surface S_2 which forms part of the conductor, and the perpendicular component is zero on the lateral surface of the tube; there would therefore only be flow of force for the exterior surface dS. If we denote by σ, the surface density of the electricity on the element dS, the total mass comprised in the volume in question is σdS. We get thus

$$F dS_1 = 4\pi\sigma dS.$$

The two surfaces being infinitely near, and parallel, we have $dS_1 = dS$, and

$$F = 4\pi\sigma.$$

Thus, *the electrical force at any point infinitely near a conductor in equilibrium, whatever be the masses in action, is equal to the electrical density close to this point multiplied by* 4π.

Since, moreover

$$F = -\frac{dV}{dn},$$

it follows that the density at the surface of a conductor may be expressed as a function of the external force, or of the potential, by the ratio,

$$\sigma = \frac{F}{4\pi} = -\frac{1}{4\pi}\cdot\frac{dV}{dn}.$$

37. CORRESPONDING ELEMENTS.—Let us lastly consider an infinitely narrow tube of force placed between the surfaces of two conductors, at which it terminates perpendicularly.

The two surface elements dS and dS', cut by the tube, are called *corresponding elements.*

Let us prolong the tube on both sides in the interior of the conductor, where we will suppose it terminated by any two surfaces. The flow of force is zero on the whole surface of the volume thus determined; for the force is tangential along the sides, and is zero at

the two ends which form part of the conductors. If σ and σ' are the densities on the corresponding elements dS and dS′, their sum is $\sigma d\mathrm{S} + \sigma' d\mathrm{S}'$. This sum must be zero, which gives,

$$\sigma d\mathrm{S} = -\sigma' d\mathrm{S}'.$$

Thus two corresponding elements contain quantities of electricity which are equal and of opposite signs.

If the two surfaces opposite each other are parallel planes, all the lines of force are perpendicular to the planes; the corresponding elements are equal, and we have

$$\sigma = -\sigma'.$$

This is also the case if the two conductors, though not plane, are infinitely near, for then the surface elements dS and dS′ may be considered equal.

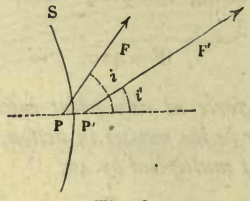

Fig. 8.

38. Uniform Field.—When the lines of force are straight and parallel, the equipotential surfaces are parallel planes. The flow of force being constant in a cylindrical tube, it will be seen that the value of the force is constant. The field is then said to be *uniform*.

Conversely, if the force in an electrical field is constant in magnitude and direction, the equipotential surfaces are necessarily parallel planes.

39. Electrified Surface Separating Two Dielectrics.—Let us suppose that an electrified surface S (Fig. 8), not belonging to a conductor, separates two different dielectrics.

Consider an elementary surface dS, on which the density is σ. Draw through the contour of this element a straight tube terminated by two equal bases, and parallel to dS, and of such a height that the lateral surface is infinitely small as compared with that of the bases. The flow of force relative to the sides of this cylinder may be neglected as compared with the flow which traverses

the bases. If F and F′ are the forces in the two media near this element, we shall have

$$4\pi\sigma dS = F'_n dS - F_n dS,$$

whence

$$4\pi\sigma = F'_n - F_n.$$

Hence, *the difference between the perpendicular components of the force on the two sides of an electrified surface is equal to the product of* 4π *by the density on the surface.*

This theorem includes that of Coulomb (36) as a particular case, in which S is the surface of a conductor.

40. Let us consider two points P and P′ in the middle of the bases of the preceding cylinder. The action F at the point P is made up of the action of all the masses external to the element, and of the action $-\phi$ of this element. In passing from P to P′ the action of the external masses does not appreciably change, but that of the element changes its sign, and becomes $+\phi$. As this force, moreover, is perpendicular by symmetry, the perpendicular component of the force varies by 2ϕ, which gives

$$F'_n - F_n = 2\phi = 4\pi\sigma,$$

whence

$$\phi = 2\pi\sigma.$$

As the perpendicular component of the force is alone modified, the tangential components of the forces F and F′ are equal on both sides the surface. If i and i' are the angles of the forces with the perpendicular on the same side of the surface, we have then

$$F' \sin i' = F \sin i.$$

The preceding equation gives, moreover,

$$F' \cos i' - F \cos i = 4\pi\sigma;$$

from which we have

$$\frac{\tan i'}{\tan i} = 1 + \frac{4\pi\sigma}{F \cos i} = 1 + \frac{4\pi\sigma}{F_n}.$$

The forces undergo therefore a sort of *refraction* on meeting an electrified surface. It may be remarked in passing, that as the law of refraction is determined by the ratio between the tangents of the angles of the perpendicular and the forces, there could never be any phenomenon analogous to total reflection.

41. Electrostatic Pressure.—Electricity forms, on the surface of a conductor, a very thin layer whose thickness it does not seem possible to determine by experiment, but which is necessarily limited. It is probable that this layer is restricted to the surface of the conductor itself, and that it occupies part of the surrounding dielectrical medium.

Let A B (Fig. 9) be the thickness of this layer. The force is zero at A on the internal surface S_1, and starting from the point B on the outer surface S_2, its value is $F = 4\pi\sigma$. In the interval the force varies from 0 to F according to an unknown law.

Let ρ be the density at a point M, and V the potential. The force at this point is perpendicular to an equipotential surface which lies between S_1 and S_2, and has the value $-\frac{dV}{dn}$. Let e be the

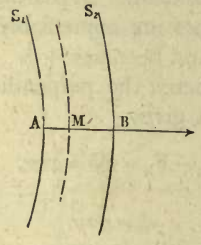

Fig. 9.

thickness AB. What has hitherto been called surface density represents the quantity of electricity on the layer e for unit surface; we have then

$$\sigma = \int_0^e \rho dn.$$

Let us take three rectangular axes, one of them in the same direction as the perpendicular at M to the equipotential surface, that is the perpendicular to the surfaces S_1 and S_2, and the two others of which, x and y, are in the tangent plane of this equipotential surface.

The second differentials $\frac{\partial^2 V}{\partial x^2}$ and $\frac{\partial^2 V}{\partial y^2}$ being zero (32), the density is expressed by,

$$\rho = \frac{1}{4\pi}\frac{d^2V}{dn^2}.$$

Hence, if p be the force on unit surface, the total force exerted on the electrical layer σdS of an element of surface is

$$p d\mathrm{S} = \int_0^e \rho d\mathrm{S} dn \left(-\frac{d\mathrm{V}}{dn} \right) = \frac{d\mathrm{S}}{4\pi} \int_0^e \frac{d^2\mathrm{V}}{dn^2} \cdot \frac{d\mathrm{V}}{dn} dn = \frac{d\mathrm{S}}{8\pi} \left[\left(\frac{d\mathrm{V}}{dn} \right)^2 \right]_{\mathrm{A}}^{\mathrm{B}}.$$

As $\frac{d\mathrm{V}}{dn}$ is zero at the point A, we have simply,

$$p = \frac{1}{8\pi} \mathrm{F}^2.$$

On the other hand,

$$\mathrm{F} = 4\pi\sigma,$$

which gives

$$p = \frac{1}{8\pi} 16\pi^2\sigma^2 = 2\pi\sigma^2.$$

It is remarkable that this expression might be obtained strictly without any other hypothesis than that of an extremely small thickness e, and therefore whatever be the law of distribution along the perpendicular.

The electricity spread upon each unit of surface is thus impelled towards the exterior with a force equal to $2\pi\sigma^2$, proportional therefore to the square of the density. This *electrostatic pressure*, or *electrical tension* is counterbalanced by the resistance of the dielectric.

When the conductors are in air, the effect of this force is to diminish the atmospheric pressure on their surface. If the initial pressure for unit surface be P in the neutral state, after electrification, it will be

$$\mathrm{P} - p = \mathrm{P} - 2\pi\sigma^2$$

at each point of the surface of the conductor.

Thus, an insulated soap bubble will increase in volume when electrified, and will resume its original volume when it is restored to the neutral state. Van Marum, for example, found that a balloon filled with hydrogen became lighter, and that its ascensional force increased, when it was electrified.

If the density is uniform on a conductor, the same is the case with the electrostatic pressure, and as this is perpendicular at every point, its resultant is zero.

If the density is variable, the electrostatic pressure is in general equivalent to a single force and to a couple. The result thus obtained

is clearly identical with that arising from a direct consideration of the action of external masses, supposed to be attached to ponderable matter, on the various electrical masses of the conductor.

42. CONSEQUENCES OF THE DISTRIBUTION OF ELECTRICITY ON THE SURFACE OF CONDUCTORS.—The preceding theorem is based on Coulomb's law considered as an experimental fact. We might follow a different course in taking as starting point this other experimental fact, which is more easily verified with strictness, that *electricity exists solely on the surface of conductors, and that in the interior there is neither electricity nor electrical force, even when the conductors contain closed cavities.*

Let us consider an electrified surface S (Fig. 10). Through a point in the interior P draw a cone $d\omega$ of infinitely small aperture,

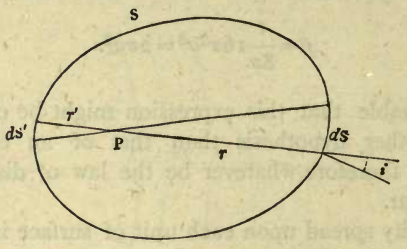

Fig. 10.

which intercepts, on the surface, at the distances r and r' two elements, dS and dS', the densities on which are respectively σ and σ'.

If the electrical forces are proportional to a function of the distance $f(r)$, the action of the element dS on unit mass at the point P may be written,

$$\phi = \sigma d\mathrm{S} f(r).$$

Calling i the angle of the radius vector r with the normal to the element dS, we have

$$r^2 d\omega = d\mathrm{S} \cos i,$$

which gives

$$\phi = \frac{\sigma d\omega}{\cos i} r^2 f(r).$$

The action of the element dS' is, in like manner,

$$\phi' = \frac{\sigma' d\omega}{\cos i'} r'^2 f(r'),$$

and these two actions are directly opposed.

If the surface in question is a sphere, the angles i and i' are equal. If, moreover, the sphere is insulated, and not exposed to any extraneous action, the distribution is homogeneous, and the densities σ and σ' are equal.

The force at the point P is zero if the actions of the opposite elements dS and dS′ are equal, and for this it is sufficient if we have the ratio

$$r^2 f(r) = r'^2 f(r') = \text{const.},$$

that is to say, that the electrical forces are inversely as the square of the distance.

The law of the square satisfies this condition, and is the only one which does. This may be shown in a simple manner by the following reasoning, which is due to M. Bertrand.

Whatever $f(r)$ may be, we may choose two values, r_1 and r_2—

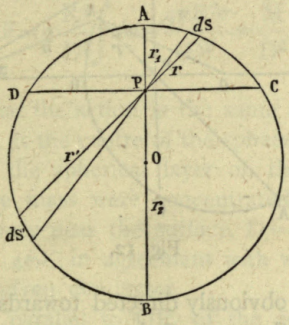

Fig. 11.

such, that between these two values of the variable the product $r^2 f(r)$ always either increases or decreases as r increases.

Let us construct a sphere (Fig. 11) whose diameter is equal to the sum $r_1 + r_2$, and let us consider the point P which divides the diameter into the two segments r_1 and r_2. The actions of the opposite elements dS and dS′, determined by the same cone of aperture $d\omega$ on this point, are: for dS,

$$\frac{\sigma d\omega}{\cos i} r^2 f(r),$$

and for dS′,

$$\frac{\sigma d\omega}{\cos i} r'^2 f(r').$$

All the values of r and r' are comprised between the limits r_1 and

r_2, and the value of r is always less than the corresponding value of r'; the value of $r^2f(r)$ for the elements of the upper portion will always be smaller or larger than that of $r'^2f(r')$ for those of the lower portion; the action of the zone DAC is therefore smaller or larger than that of the zone DBC. Equilibrium would thus be impossible unless $r^2f(r)$ is a constant—that is, unless the function $f(r)$ were exactly in the inverse ratio of the square of the distance.

43. ACTIONS OF SPHERICAL LAYERS.—*The action of a homogeneous spherical layer on an external point is the same as if the whole mass were concentrated at the centre of the sphere.*

For let us consider the action which a sphere S (Fig. 12), covered with a homogeneous layer of density σ, exerts upon an external point P.

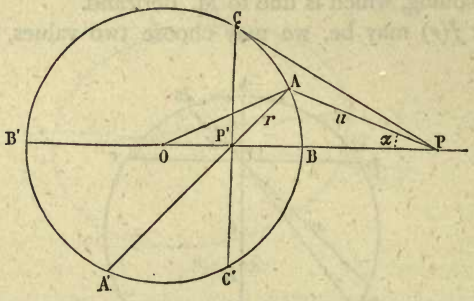

Fig. 12.

The action being obviously directed towards the centre, it will be sufficient if we take the sum of the components of the elementary actions in this direction.

This component for the surface element dS at the point A, which is at a distance ρ from P, is

$$\phi = \frac{\sigma d\mathrm{S}}{\rho^2} \cos \alpha.$$

Let P′ be the conjugate of the point P; that is to say, such that

$$\mathrm{OP'}.\mathrm{OP} = \mathrm{R}^2;$$

if we draw AP′ and AO, the triangles AOP′ and AOP are similar, for the angle at O is common, and we have the ratio

$$\frac{\mathrm{OP'}}{\mathrm{R}} = \frac{\mathrm{R}}{\mathrm{OP}};$$

hence the angles OPA and OAP′ are equal; calling r the distance P′A, and D the distance OP, we have

$$\frac{\rho}{r} = \frac{D}{R}.$$

Lastly, let $d\omega$ be the angle which subtends the element dS, seen from the point P′,

$$r^2 d\omega = d\mathrm{S} \cos \alpha.$$

Replacing dS cos α by this value in the expression for the component ϕ, we get

$$\phi = \sigma d\omega \frac{r^2}{\rho^2} = \sigma d\omega \frac{R^2}{D^2},$$

and the total action of the sphere is

$$F = \sigma \frac{R^2}{D^2} \int d\omega = \frac{4\pi R^2 \sigma}{D^2} = \frac{M}{D^2}.$$

It is thus apparent that the action is the same as if the whole mass M were concentrated at the centre of the sphere.

The potential of the spherical layer on the outside is also the same as if the whole mass were concentrated at the centre.

If the point P is very near the surface, D = R and the action of the layer is equal to $4\pi\sigma$, in agreement with what we have already found (36) for any given conductor.

The component, parallel to OP, of the action exerted by the surface element dS, only depends on the angle $d\omega$ under which this element is seen from the point P′. This component is then the same for an element dS′ situate at A′, and opposed to the former in reference to the point P′.

The same is the case for all the elements of the zone CB′C′ compared in pairs with the elements of the layer CBC′.

The plane CC′ *thus divides the surface of the sphere into two parts, whose actions on the point* P *are equal, each of them being equal to* $2\pi\sigma \frac{R^2}{D^2}$.

If the point P moves to an infinite distance, the two zones tend to become equal; if it is infinitely near the surface, the anterior zone becomes infinitely small, and its action upon an infinitely near point is reduced to $2\pi\sigma$. We have already obtained this result (40) for any given surface.

44. Action of a Sphere Consisting of Homogeneous Layers.—Let us first consider a sphere electrified throughout its whole mass, and made up of homogeneous concentric layers.

The action of this sphere on an external point is the same as if the whole mass were concentrated at the centre, or were carried to the surface so as to form a uniform layer.

On a point in the interior of the sphere the action of the layers which surround it is null; that of the layer whose radius is less than its distance from the centre is still the same as if the mass were concentrated at the centre.

Proceeding towards the centre the acting mass diminishes then more and more; the direction of the force is always along the radius, and depends on the manner in which the density varies.

When the density ρ of the sphere is constant, the action exerted on a point in the interior, at the distance r, is equal to

$$F = \frac{1}{r^2} \cdot \frac{4}{3} \pi r^3 \rho = \frac{4}{3} \pi \rho r ;$$

it is thus proportional to the distance from the centre.

If the density at a point is proportional to the n^{th} power of its distance l from the centre, $\rho = al^n$; the action of the layer dl at the distance r is

$$\frac{4\pi l^2 a l^n dl}{r^2} = \frac{4\pi a}{r^2} l^{n+2} dl,$$

and the total action

$$F = \frac{4\pi a}{r^2} \int_0^r l^{n+2} dl = \frac{4\pi a r^{n+3}}{r^2 (n+3)} = \frac{4\pi a}{n+3} r^{n+1}.$$

This action is constant for $n = -1$; it increases, on the contrary, in approaching the centre, if $n < -1$.

The total mass of the sphere of radius R is

$$M = \int 4\pi l^2 a l^n dl = \frac{4\pi a}{n+3} R^{n+3},$$

which gives

$$F = M \frac{r^{n+1}}{R^{n+3}}.$$

Let us suppose that the density varies according to an arbitrary law. Let m be the mass external to the sphere which passes through

the point P, ρ_0 the mean density of the whole sphere, and ρ the mean density of the external layers, we shall have

$$F=\frac{M-m}{r^2}=\frac{1}{r^2}\left[\frac{4}{3}\pi R^3\rho_0-\frac{4}{3}\pi(R^3-r^3)\rho\right]=\frac{M}{r^2}\left[1-\left(1-\frac{r^3}{R^3}\right)\frac{\rho}{\rho_0}\right].$$

The force at the surface is $F_1=\frac{M}{R^2}$, which gives

$$\frac{F}{F_1}=\frac{R^2}{r^2}\left[1-\left(1-\frac{r^3}{R^3}\right)\frac{\rho}{\rho_0}\right].$$

The force may at first be an increasing one on starting from the surface, then attain a maximum, and then go on decreasing to the centre. This, for instance, is the case with the variations of gravity in the interior of the globe. For this we must have

$$\left(1-\frac{r^3}{R^3}\right)\frac{\rho}{\rho_0}<1-\frac{r^2}{R^2}.$$

If the thickness $h=R-r$ is very small as compared with R, this condition may be written

$$3\left(\frac{r}{R}\right)^2\frac{h}{R}\frac{\rho}{\rho_0}<2\frac{r}{R}\cdot\frac{h}{R}, \quad \text{or} \quad \frac{\rho}{\rho_0}<\frac{2}{3}<0{\cdot}67.$$

The mean density of the Earth being about equal to 5·5, and that of the surface to 2·5, we have, in fact,

$$\frac{\rho}{\rho_0}=\frac{2{\cdot}5}{5{\cdot}5}=0{\cdot}45.$$

CHAPTER III.

GENERAL THEOREMS.

45. EMISSION AND ABSORPTION OF FORCE BY ELECTRICAL MASSES.—The flow of force through an orthogonal tube remains constant, as we have seen (35), so long as the tube does not encounter any acting mass; the direction of the transmission is that in which the potential diminishes.

If the tube encounters a mass of electricity m, the flow of force experiences an increase $4\pi m$ at the boundary, which it retains beyond it as long as the tube does not encounter fresh masses. If the mass thus encountered is on the surface of a conductor, it is such as will reduce to zero the flow of force transmitted by the tube. We may then say that of two corresponding elements dS and dS′, whose densities are σ and σ', the positive electricity of the element dS *emits* a flow of force $4\pi\sigma d$S, which at the other end of the tube is *absorbed* in the equal quantity of negative electricity on the element dS′.

On Faraday's views there are no unlimited tubes of force. A tube proceeding from an electrified body would always terminate somewhere on another body, so as to induce on the corresponding element an equal quantity of electricity of the opposite sign. On this view, there could nowhere be an absolute and independent quantity of electricity which has not its complementary quantity at the other end of the tube.

No line of force can exist between two points charged with the same electricity. Nor can there be any between two points at the same potential. In fact, no line of force can correspond on a conductor to a point not charged with electricity (36).

46. THE POTENTIAL CAN NEITHER HAVE A MAXIMUM NOR A MINIMUM OUTSIDE THE ACTING MASSES.—An insulated mass m concentrated in a point may be regarded as a layer spread over a very small conductor. If the mass m is positive, the point A, which

it occupies, is a centre of the emission of force in all directions; if it is negative, it is a centre of absorption for all directions. In both cases the adjacent equipotential surfaces are closed surfaces. In the limit the surfaces are spherical even; for at a small distance r from the point A, the potential of the external masses may be neglected as compared with the potential $\frac{m}{r}$ of the mass m.

The equipotential surfaces being closed about A, the potential has a maximum or minimum value; maximum, if the mass m is positive; minimum, if it is negative.

Conversely, there is electricity wherever the potential is maximum or minimum.

For, starting from a point A of maximum, the potential decreases in all directions, the adjacent equipotential surfaces are necessarily closed, and the flow of force which traverses, for example, a very small spherical surface comprising the point A, has a finite value Q. In the interior of this surface there is therefore a quantity of positive electricity equal to $\frac{Q}{4\pi}$.

In like manner, a minimum of potential is a centre of absorption of force, where there is a corresponding mass of negative electricity.

Hence, *in any given electrical system, there can neither be absolute maximum nor absolute minimum of potential outside the acting masses.*

47. POINTS AND LINES OF EQUILIBRIUM.—Let V_0 be the potential at a fixed point P_0. The potential V at an adjacent point P, whose co-ordinates referred to axes passing through the former are x, y, z, may be expressed as a function of increasing powers of the co-ordinates,

$$V = V_0 + H_1 + H_2 + H_3 + \ldots\ldots,$$

H_1, $H_2 \ldots . H_n$ being homogeneous functions of the first, of the second.... of the n^{th} degree of the co-ordinates.

As the equation of Laplace should be satisfied separately by each of these functions, we shall have in general

$$\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial^2 H_n}{\partial y^2} + \frac{\partial^2 H_n}{\partial x^2} = 0.$$

If the function H_1 is identical with zero, that is to say, if the three partial differentials of the potential

$$\left(\frac{\partial V}{\partial x}\right)_0, \quad \left(\frac{\partial V}{\partial y}\right)_0, \quad \left(\frac{\partial V}{\partial z}\right)_0,$$

are null at the point P_0, this point is a singular point of the level surface V_0; the force there is null, it is a *point of equilibrium.*

For adjacent points we may neglect the powers of the co-ordinates higher than the second, and the expression of the potential V reduces to

$$V = V_0 + H_2.$$

The equation

$$H_2 = 0$$

represents a cone of the second degree tangential to the equipotential surface at the point of equilibrium P_0.

If the function H_2 is itself identical with null, as well as some of the following ones, and if H_n is the first function of the development, which does not vanish, the equation

$$H_n = 0$$

will represent, in like manner, a cone of the n^{th} degree, tangential to the equipotential surface at the point P_0.

This cone will be formed of n sheets, or of a smaller number, corresponding to an equal number of sheets of the level surfaces.

If the sheets of the cone do not intersect, neither do the equipotential surfaces, and P_0 is an isolated point of equilibrium.

If the sheets of the cone do intersect, every line of intersection is tangential to the intersection of the corresponding sheets of the equipotential surface—that is to say, a *line of equilibrium* which passes through the point P_0.

48. *If the equipotential surface at the point* P_0 *consists of two sheets which intersect, the intersection of the sheets takes place at a right angle.*

When the equipotential surface consists of two sheets which intersect, the cone of the second degree, which is tangential at a point P_0 of the line of intersection, reduces to two planes. If we take the tangent to this line for the z axis, the equation of the cone $H_2 = 0$ will contain no term in z.

In order to satisfy Laplace's equation, which reduces to

$$\frac{\partial^2 H_2}{\partial x^2} + \frac{\partial^2 H_2}{\partial y^2} = 0,$$

the coefficients of the terms in x^2 and in y^2 must be equal and of

opposite signs, so that the equation of the cone is of the form

$$A(x^2 - y^2) + Bxy = 0;$$

they represent two rectangular planes.

Let us consider, as an example, the surface of a conductor charged partly with positive electricity and partly with negative; the line of separation of the two layers is a *neutral line.* The force is null in all points of the neutral line (35), and there is in the dielectric another equipotential surface at the same potential as the conductor, which cuts it perpendicularly along this line. It will be remarked that this particular equipotential surface separates the lines of force, which start from the conductor, from those which terminate there. It might, therefore, be considered as a limiting surface of the lines of force.

49. *If the equipotential surface consists of n sheets which intersect along the same line, the successive intersections take place at the same angle* $\frac{\pi}{n}$.

Starting from a point P_0 of the line of equilibrium, all the functions H of the development of the potential, up to that of the degree n, are identical with zero, since the tangent cone consists of n sheets. In order that the equation $H_n = 0$ shall represent n planes passing through the z axis, it must not contain any terms in z, and Laplace's equation reduces to

$$\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial^2 H_n}{\partial y^2} = 0.$$

If r denotes the distance of a point P from the z axis, and putting

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

Laplace's equation becomes

$$\frac{\partial^2 H_n}{\partial r^2} + \frac{1}{r}\frac{\partial H_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_n}{\partial \theta^2} = 0.$$

The function of the degree n, which satisfies this equation, is

$$H_n = A_n r^n \cos(n\theta + \alpha_n).$$

Making it equal to zero, we get

$$\cos(n\theta + \alpha_n) = 0,$$

an equation which represents n planes passing through the z axis, and the successive angles of which are equal to $\frac{\pi}{n}$.

50. THERE IS ONLY ONE STATE OF EQUILIBRIUM.—It may be observed, in the first place, that *the superposition of two states of equilibrium is itself a state of equilibrium.*

For in each of the two states of equilibrium the potential is constant on all the conductors. The superposition of the two systems of electrical layers produces, at each point, a potential equal to the sum of the potentials relative to the two primitive states. The potential is constant, therefore, on each of the conductors, and equilibrium exists.

It follows from this, that if we change the electrical density at each point in a constant ratio, a new state of equilibrium will be formed, for the operation amounts to superposing two or more identical states of equilibrium.

51. *A system of conductors* A_1, A_2, $A_3 \ldots$, *whose electrical charges are separately null, is necessarily in the neutral state.*

Let V_1, V_2, V_3 denote the potentials of these various conductors, and let V_1 be the greatest.

There can be no point in the dielectric where the potential is higher than V_1, since there is no maximum of potential outside the acting masses. The potential sinks, therefore, in all directions from the conductor A_1; all the lines of force start from this conductor, and none terminate there. As the sum of the flows of force must be zero (for by hypothesis the total charge of A_1 is zero) it is seen that all the elementary flows of forces are zero. The density is therefore zero over the whole surface, and therefore the conductor is not electrified.

The conductor A_1, being in the neutral state, may be suppressed, and the same reasoning applied to the next conductor; it may thus be shown successively that all the conductors are in the neutral state.

The conductors A_1, A_2, $A_3 \ldots$, having charges M_1, M_2, $M_3 \ldots$, differing from zero, let us now suppose that two states of equilibrium are possible, such that the densities on A_1, A_2, A_3 are σ_1, σ_2, $\sigma_3 \ldots$ in the first case, and σ'_1, σ'_2, $\sigma'_3 \ldots$, in the second.

Changing the signs of all the electrical masses of the second state,

there would still be a state of equilibrium, which, superposed on the first, will give a new state of equilibrium, in which the total charge will be zero on each of the conductors.

In this case, from the preceding observation, the density must be everywhere zero. We have, then,

$$\sigma_1 = \sigma'_1,$$
$$\sigma_2 = \sigma'_2,$$
$$\sigma_3 = \sigma'_3,$$
$$\ldots\ldots$$

The distribution is therefore the same in both cases, and therefore the equilibrium is singular. If the proposed system comprised fixed masses, they might always be supposed to be upon infinitely small conductors, and the reasoning would not be changed. The theorem is therefore a general one.

52. THEOREMS RELATING TO CLOSED SURFACES.—*If the potential is constant on a closed surface* S, *not containing any acting mass, it is constant throughout the whole interior.*

The potential, in fact, could not vary in the interior of the surface S without attaining at one point a maximum or minimum value, which is impossible, as there is no electricity (46).

If the surface in question is the external surface of a conductor, it is seen that the potential is constant not only in the mass of the conductor, which is a necessary consequence of the conditions of equilibrium, but also in the cavities which this may contain.

53. *If a surface at constant potential comprises a portion of a dielectric, the potential is constant not only in the interior of this surface, but also in all the exterior space outside the acting masses.*

For, as the potential is constant in the interior, no line of force can traverse the surface; all those which could meet it are either produced there or are absorbed—which is impossible, since there is no electricity on the surface. Hence no line of force meets the surface.

As no line of force meets the surface, the potential is constan on the outside close to it; hence a new surface S′ may be drawn having the same property, to which the same reasoning may be applied, and so on indefinitely, provided we always keep outside the acting masses.

54. This latter theorem was demonstrated by Gauss in a different manner, by means of the following lemma:—

If a spherical surface contains no acting mass, the potential at the

centre is the mean of the values of the potential at the different points of the surface.

For let R be the radius of the sphere, m one of the masses at a point A, at a distance d from the centre, and r the distance of a surface element dS from the point A; the mean value, on the surface, of the potential due to the mass m, is

$$V_m = \frac{\int \frac{m d\mathrm{S}}{r}}{4\pi \mathrm{R}^2};$$

the sum $\int \frac{m d\mathrm{S}}{r}$ is the value at A, of the potential of a homogeneous layer of density m, which would cover the sphere; it is equal (43) to

$$\frac{4\pi \mathrm{R}^2 m}{d}.$$

The mean value of the potential on the sphere is then

$$V_m = \frac{m}{d},$$

that is to say, equal to the value of the potential at the centre.

This reasoning manifestly extends to any number of masses.

That being admitted, let us suppose that in a portion of the dielectric bounded by the surface S, the potential has a constant value V; if the value of the potential were different from V on the outside, it would always be possible to draw a sphere having its centre in the inside of S, and on the outside only meeting points where the potential would be always either greater or less than V; but, in that case, the mean value of the potential on the sphere would be different from its value at the centre. The potential on the exterior of the surface S cannot, therefore, be different from V.

55. *When a closed surface* S *surrounds all the acting masses, and the potential on this surface has a constant value* V, *at each of the external points there is a value comprised between* V *and zero.*

Let us suppose V positive. As the potential is zero at an infinite distance, it cannot have a higher value than V, at a point P external to the surface, without there is somewhere a maximum potential, and therefore electrical masses, which is contrary to the hypothesis. In like manner the potential at P cannot be less than V, for otherwise there would be a minimum. The potential on

the outside is therefore between V and zero. It cannot be equal to V, unless V itself is zero. In fact, it cannot be equal to V without being a maximum in reference to adjacent points, or without forming part of a space at constant potential which would extend to the surface S; but, from the preceding theorem, the potential in this latter case would be constant throughout the dielectric, and could only have the value zero, for it is zero at an infinite distance.

56. *A conducting surface* S, *which contains all the acting masses, can only have electricity of one kind.*

For let V be the potential of the surface, which we will suppose is positive. If there were negative electricity at a point A, lines of force would terminate there, and there must be somewhere an external point where the potential has a higher value than V, which is impossible from the preceding theorem.

57. *When, in a system in equilibrium, a conductor envelopes diverse masses of electricity, the algebraical sum of the quantities of*

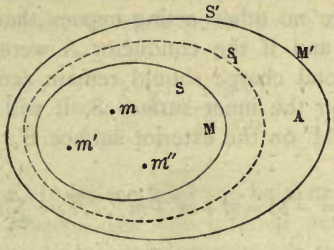

Fig. 13.

electricity in the interior, and on the internal surface of the body, is zero.

Let m, m', m''... be the masses comprised within the interior of the conductor A (Fig. 13), and M the mass of the layer spread on the internal surface S; to which we may add that there may be electricity on the external surface S', and other masses beyond. On a closed surface S_1, in the conductor, and comprising all the internal masses, the force is null at each point.

The flow of force relative to this surface is therefore null, and accordingly the algebraical sum of the masses which it contains is null. We have then

$$M + m + m' + m'' + \ldots\ldots = 0,$$

whence

$$M + \sum m = 0.$$

The layer M, developed by induction on the surface S, is equal and of opposite sign to the algebraical sum of the masses comprised within the cavity. This layer *absorbs* the flow of force emanating from the internal masses. The system of this layer, and of the masses which it comprises, gives a potential zero, and its action is zero at every outside point.

If the conductor A is in connection with the earth, its potential is zero; if then there are no other acting masses than those contained within the cavity, the external surface S′ is in the neutral state, and the potential is everywhere zero outside the cavity.

An electrical mass M′, placed on the insulated surface S′, acquires there a distribution independent of the internal masses. It will produce a constant potential V′ throughout the whole extent of the conductor A, and of the cavities which it encloses; and this potential will add itself at each point to the potential already existing. The mass M′ will not exert there any influence on the equilibrium of the internal masses.

58. If there are no other acting masses than those comprised within the cavity, and if the conductor A were originally in the neutral state, its total charge should remain zero; simultaneously with the layer M on the inner surface S, it will develop an equal and opposite layer M′ on the exterior surface S′, and we shall have

$$M' = -M = m + m' + m'' \ldots\ldots = \sum m.$$

This layer M′, equal to the algebraical sum of the internal masses, and of the same sign, is of itself in equilibrium whatever be the position of the internal masses.

We get thus Faraday's law: *The quantity of electricity induced by an electrical system on a conductor, which surrounds it, is equal to the quantity of inducing electricity.*

59. *The action which given electrical masses exert on the exterior of any closed surface, is the same as that of a layer of the same mass spread on this surface according to a certain law.*

Let m, m', m'', ... or $\sum m$, be the given masses, which we will suppose fixed as if they belonged to non-conducting bodies; a surface S of any given form envelopes them. Suppose that, for a moment, we replace this surface by a material sheet, forming an infinitely thin conductor in connection with the earth; the internal surface of this sheet will become covered with a mass $-M = -\sum m$, whose potential, for all external points, is equal and of opposite sign to

that of $\sum m$. A layer $+M$, distributed in the same way on the surface S, will have everywhere on the outside a potential equal, and of the same sign, to that of the masses in question, $\sum m$.

The layer M will not in general be in equilibrium of itself; that is to say, that it will not have the distribution which would result from the form of the surface, and from the action of external masses; the lines of force will not cut it perpendicularly.

60. *The layer* $+M$ *will be in equilibrium of itself, if the surface* S *is an equipotential surface of the primitive system, taking into account the external masses.*

Let us denote by $\sum m'$ the external masses. The lines of force of the field are the same at the exterior of S for the system $\sum m$ and $\sum m'$, and for the system M and $\sum m'$; these lines being, by supposition, normal to S, the layer M which covers it is in equilibrium, and has a constant potential V.

We may then, for external points, replace the system $\sum m$ by an equal mass in equilibrium on an equipotential surface which surrounds it, the density at each point being determined by the condition

$$\sigma = \frac{F}{4\pi} = -\frac{1}{4\pi}\frac{dV}{dn}.$$

This substitution always modifies the field in the interior of the surface S; in the present case the internal potential has become constant and equal to V, for it is constant on the surface, and the cavity no longer contains electricity.

If the potential is constant, the force is zero; the external system $\sum m'$, and the layer M, exert equal and contrary actions at each point of the interior of S; a layer $-M$ would exert actions equal to, and of the same sign as, that of $\sum m'$. Thus for all the internal points of the equipotential surface S, we may replace the action of external masses by that of a layer in equilibrium, equal to the internal masses, and of the opposite sign.

61. The following theorems follow from this: If we consider an equipotential surface S in any electrical system, we may—

1st. *For all external points replace the internal masses by a mass* M, *equal and of the same sign, in equilibrium on this surface;*

2nd. *For points in the interior, we may replace the external masses* $\sum m'$ *by the same mass* M *with changed sign; that is to say, by a mass equal and opposite to the internal masses, this layer being still in equilibrium.*

62. GAUSS' THEOREM.—Given two electrical systems, one consisting of the masses m_1, m_2, m_3, ... and producing a potential V, the other of masses m'_1, m'_2, m'_3, ... and producing a potential V', we shall have the identical equation:

$$\sum mV' = \sum m'V;$$

that is to say, that *the sum of the elementary masses of the first system, multiplied respectively by the value of the potential of the second system at the point which they occupy, is equal to the corresponding sum relative to the masses of the second system;* the summations must be replaced by integrals if the masses occupy a finite extent.

This proposition is an identity; to see this, we need only replace the potentials by their values as functions of the masses, and of the distances. It then appears that each member of the equation is equal to the sum of the products obtained by multiplying each mass of one system by a mass of the other, and dividing the product by the distance separating them.

It will be sufficient to remark that the sums $\sum m'V$ and $\sum mV'$ denote the work which must be expended to bring them respectively in presence of each other, from an infinite distance to the positions which they occupy, the two systems $\sum m$ and $\sum m'$ being supposed rigid; and in either case the work is evidently the same.

When the two systems we are considering are conductors in equilibrium, A_1, A_2, A_3, the potential is constant on each conductor, and if we denote their total masses by M_1, M_2, M_3....., and M'_1, M'_2, M'_3....., the equation becomes

$$M_1V'_1 + M_2V'_2 + M_3V'_3 \ldots\ldots = M'_1V_1 + M'_2V_2 + M'_3V_3 \ldots\ldots$$

63. The following theorems may be considered as corollaries of that of Gauss, as has been shown by M. Bertrand:—

I.—*If a conductor* A *in the neutral state, whether insulated or not, is exposed to the action of an electrical mass m placed successively at two points* P *and* P' *of the dielectric, the potential due to the induced charge at* A, *will be the same at the point* P' *in the first case, as at the point* P *in the second.*

Let us observe, in the first place, that any point of the dielectric may always be considered as the centre of an infinitely small conducting sphere, for the charge which this sphere acquires is constantly zero, and always produces at its centre a potential equal to zero.

If the body is connected with the earth, we shall have to consider the two follow ng states of equilibrium :—

		Potentials.	Charges.
1st.	Sphere P	U	m
	Sphere P′	V	0
	Body A	0	x
2nd.	Sphere P	V′	0
	Sphere P′	U′	m
	Body A	0	x'

Applying Gauss' theorem to these two conditions, we get

$$mV' = mV,$$

whence

$$V = V'.$$

If the body A is insulated, its potential is not equal to zero, but its charge is null in both states, and the final result is the same.

II.—*If each of the two conductors* A *and* B *is successively put in connection with a source which raises it to potential* V, *the other being in connection with the earth, and therefore at zero potential, the quantity of electricity developed by induction on the latter is the same in both cases.*

We have, in fact, in the first case,

	Potential.	Charge.
A	V	x
B	0	$-M$

and in the second,

	Potential.	Charge.
A	0	$-M'$
B	V	x

Applying the preceding theorem, we get

$$M'V = MV,$$

whence

$$M' = M.$$

64. EARNSHAW'S THEOREM.—*An electrified body cannot be in stable equilibrium in an electrical field.*

Let an electrified body A be placed in a field produced by external masses B, and let us suppose all the masses fixed, including that of A.

Let m denote the electrical mass of a volume element of A at a point P, where the potential of the external masses is V. The energy of the body in the field is

$$W = \sum mV. \tag{1}$$

For stable equilibrium, the differential $\frac{\partial W}{\partial r}$ must be null or positive in any given direction r.

Let x, y, z be the co-ordinates of the point P; ξ, η, ζ those of a point P_0 taken in the interior of the body A, and a, b, c that of the point P with reference to three new axes parallel to the first, and passing through the point P_0; we shall have

$$x = a + \xi,$$
$$y = b + \eta,$$
$$z = c + \zeta.$$

The potential V may be considered as a function of a, b, c, and of ξ, η, ζ. If the body A is constrained to move parallel to itself, we shall have

$$dx = d\xi,$$
$$dy = d\eta,$$
$$dz = d\zeta,$$

and therefore,

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \zeta^2}. \tag{2}$$

As this sum is zero for each of the terms mV of the second member of the equation (1), we shall also have

$$\Delta W = \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} + \frac{\partial^2 W}{\partial \zeta^2} = 0.$$

The energy W is therefore a function of the co-ordinates of the point P_0, and this function satisfies Laplace's equation, as long as equation (2) itself is satisfied. We may assume that the point P_0 is comprised within a sphere of radius r, so small that the body in question, A, does not meet any external masses. The variation of energy for any displacement parallel to one of the radii of this sphere, will be

$$dW = \frac{\partial W}{\partial r} dr.$$

Applying Green's equation to the surface of this sphere, we shall have

$$\int \frac{\partial W}{\partial r} dS = \int \Delta W dv = 0.$$

As the integral $\int \frac{\partial W}{\partial r} dS$ becomes null, it follows that the differential $\frac{\partial W}{\partial r}$ is negative for certain directions, and positive for others; hence, the body A is not in equilibrium, and tends to move towards places for which the energy W diminishes.

There is equilibrium if $\frac{\partial W}{\partial r}$ is always zero; that is to say, if the energy is constant, or passes through an absolute maximum or minimum. We have then

$$\frac{\partial W}{\partial \xi} = \frac{\partial W}{\partial x} = \sum m \frac{\partial V}{\partial x} = -\sum mX = 0,$$

and in like manner,

$$\sum mY = 0,$$

$$\sum mZ = 0.$$

The components X, Y, Z of the force of the field, produced by the external masses, may be developed as a function of increasing powers of the co-ordinates *a*, *b*, *c*, and of the components X_0, Y_0, Z_0, relative to the point P_0, which gives

$$X = X_0 + H_1 + H_2 + \ldots\ldots + H_n + \ldots\ldots,$$

H_1, H_2, ... H_n.. being functions of the 1, 2.... *n*.. degrees of the co-ordinates *a*, *b*, and *c*.

We get then

$$\sum mX = \sum m (X_0 + H_1 + H_2 + \ldots\ldots + H_n) = 0.$$

To satisfy this equation, all the coefficients of the development must be zero. That necessitates, in the first place, that all the differentials of X be null; then, that the position of the point P_0 be arbitrary; and, lastly, that we have

$$X_0 \sum m = 0.$$

Hence, either the total mass $\sum m$ of the body A must be zero, or the components X_0, Y_0, and Z_0 of the force must themselves be zero.

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In order, therefore, that an electrified body be in equilibrium in an electrical field, either the force of the field must be zero, or the field must be uniform, with the condition that the algebraical sum of the electrical masses which the body possesses, be zero.

In this demonstration, we have assumed that the electricity was fixed on the body A, and that the external masses themselves formed a rigid system. The theorem applies with the more reason to the case in which the system contained conductors. If stable equilibrium does not exist when connections are introduced into the system, this is still less the case when these connections are suppressed; for instance, when the electrified bodies are in part conductors, which leaves more play to the displacement of electrical masses.

CHAPTER IV.

ELECTRICAL EQUILIBRIUM.

65. CONDITIONS OF THE EQUILIBRIUM OF CONDUCTORS.—The general problem of electrical equilibrium, when restricted to the case of conductors, may be thus enunciated :—

With conductors of given shape and position, one set insulated, and the other in connection with the earth, the former being charged with a definite quantity of electricity, it is required to determine the potential at each point.

In other words, the problem amounts to determining a function V of co-ordinates, which satisfies the following conditions :—

1st. The function must be zero at an infinite distance, and must have a constant value on each of the conductors, this value being zero on all conductors in communication with the earth.

2nd. The sum of the three second partial differentials must be zero over the whole surface of the dielectrics and in the interior of conductors, for the electrical density is zero at all these points.

3rd. At every point of the surface of the conductors, the density is determined by the equation

$$\sigma = -\frac{1}{4\pi}\frac{d\mathrm{V}}{dn},$$

so that the total charge upon one of the conductors is expressed by

$$\mathrm{M} = \int \sigma d\mathrm{S} = -\frac{1}{4\pi}\int \frac{d\mathrm{V}}{dn} d\mathrm{S}.$$

These three conditions are sufficient, for they determine a state of equilibrium which satisfies the data of the question, and only one state of equilibrium is possible for the system.

This problem frequently presents great difficulties from the mathematical point of view, and no general solution is known. It has only been completely solved for a few special cases, the most

important of which we shall afterwards investigate; but we may deduce from the general enunciation a certain number of remarkable properties.

66. General Observations.—Let us first suppose that among all the conductors which are in presence of each other, A_1, $A_2 \ldots A_n$, one of them, A_1, has received a charge M, and that all the others are in connection with the earth.

When equilibrium has been established, the potential has a constant value V_1 on A_1, and it is zero on all other conductors. Let us suppose $V_1 > 0$.

Within the whole dielectric the potential can neither be higher than V_1 nor lower than zero, and it lies between V_1 and 0 (55). It follows from this, that those conductors connected with the earth only possess negative electricity; for if there were positive spaces on their surface, lines of force would start from them towards the points where the potential was lower, that is to say negative, and these points do not exist.

All the lines of force of the field start then exclusively from the conductor A_1; one set terminates in the conductors in connection with the earth, the others proceed towards an infinite distance.

It follows from this that the negative charge on these conductors is only a fraction of that upon A_1; the two charges would only be equal, provided one of the conductors in connection with the earth completely surrounded A_1.

67. Let us now suppose that in the vicinity of A_1 there are other insulated conductors A_2, A_3, ..., at first in the neutral state, and whose total charge therefore is zero.

The potential is still positive and is less than V_1 within the whole dielectric. It has a constant value on each of the other conductors; this value is positive, since part of the surface of each of the conductors is charged with positive electricity, and therefore lines of force start from them, and these lines of force proceed towards spaces where the potential is everywhere positive.

Let A_2 be that insulated conductor whose potential V_2 is highest; part of its surface is negative, it therefore receives lines of force. None of these lines of force come to it from the earth, nor, by hypothesis, from the other conductors whose potential is lower; they all proceed then from the conductor A_1, and therefore V_2 is less than V_1. As, moreover, the lines of force received by A_2 do not form the whole of those proceeding from A_1, each of the positive and negative layers which make up the zero charge of A_2 is smaller than the total charge of A_1.

The same reasoning applies to all the other conductors; thus taking them in decreasing magnitude of potential, A_3, for instance, receives lines of force from A_1 and A_2, and these latter may be considered as proceeding indirectly from A_1. On each of the insulated conductors the negative charge is less than the positive of A_1, provided that none of them forms a closed surface completely surrounding the conductor A_1.

68. Relation of Charges to Potentials.—Denoting by A_1, A_2, ... A_n the conductors, let M_1, M_2, ... M_n be the respective charges, and V_1, V_2, ... V_n the corresponding potentials.

Let us first suppose all the conductors insulated, in the neutral state, and at zero potential. If we give unit positive charge to one of them, A_1, its potential becomes α_{11}, and those of the other conductors are respectively α_{21}, α_{31}, ... α_{n1}. If instead of unit charge, the charge M_1 be given to A_1, all the potentials would be multiplied by M_1; they would be $\alpha_{11} M_1$, $\alpha_{21} M_1$, ... $\alpha_{n1} M_1$. Let us suppose that A_1 is discharged, and that we give the charge M_2 to A_2, the potentials will become $\alpha_{12} M_2$, $\alpha_{22} M_2$ ..... $\alpha_{n2} M_2$; and so forth. Now the final state, when all the conductors receive their respective charges simultaneously, is that in which all the states, obtained thus in succession, are superposed; to express then the potential of each conductor we shall have an equation of the form

$$(1) \qquad V_p = \alpha_{p1} M_1 + \alpha_{p2} M_2 ..... \alpha_{pn} M_n,$$

and, therefore, n similar equations for the whole system.

From this the following theorem is deduced.

In any electrical system in equilibrium, the potentials of the several conductors may be expressed as a linear function of the charges.

Among the n^2 coefficients of the equation (1), the coefficient α_{pp} expresses the potential of the conductor A_p when it is charged with unit electricity, all the others being in the neutral state; a coefficient such as α_{qp} denotes the potential, which a conductor such as A_q would acquire in the same time.

It is easy to see that these latter coefficients satisfy the relation

$$(2) \qquad \alpha_{pq} = \alpha_{qp}.$$

For, let us consider the two successive states in which each of the conductors A_p and A_q is alone charged with unit electricity, all the others being in the neutral state; applying Gauss' theorem (62) the relation (2) is at once obtained.

The remark made above (67) shows that all the coefficients α are positive, and that a coefficient such as α_{qp} is never greater than α_{pp} or α_{qq}.

69. If we solve equations (1) in reference to the charges, we shall have n equations of the form

$$M_p = \gamma_{p1}V_1 + \gamma_{p2}V_2 \ldots\ldots + \gamma_{pn}V_n, \tag{3}$$

containing n^2 coefficients, the signification of which is at once manifest.

The coefficient γ_{pp} expresses the charge which must be given to the conductor A_p to raise it to *unit* potential, all the others being at zero potential. This coefficient, which plays a conspicuous part in the theory of electricity, is called the *capacity* of the conductor A_p; we shall revert to it in a moment. A coefficient such as γ_{qp} expresses the charge acquired by the conductor A_q in connection with the earth; it might be called the coefficient of electricity induced by A_p upon A_q.

The application of Gauss' theorem in the case of two successive states, in which each of the conductors A_p and A_q is raised to *unit* potential, the others being in communication with the earth, shows that these coefficients are also equal in pairs, and that we have the ratio

$$\gamma_{pq} = \gamma_{qp}, \tag{4}$$

which is only an extension of the theorem demonstrated above (63) for two conductors.

If we refer to the observation in 66, it is easy to see that while the coefficients γ_{pp}, which express the capacities, are all positive, the coefficients of the induced electricity, such as γ_{pq}, are all negative; moreover, that the sum of all those which relate to the induction exerted by the same conductor, is never higher in absolute value than the capacity of this conductor itself. For instance, we have, necessarily,

$$\gamma_{pp} > -[\gamma_{1p} + \gamma_{2p} \ldots\ldots + \gamma_{np}],$$

unless one of the conductors in connection with the earth—A_q, for instance—completely envelopes the conductor A_p. In this case, we should have

$$\gamma_{pp} = -\gamma_{qp},$$

and the $n-2$ other coefficients relative to the conductor A_p, γ_{1p}, γ_{2p} ..... γ_{np}, would be null.

70. Analogies of the Problem of Electrical Equilibrium. —It is interesting to compare with the problem, which we have just treated, two other problems relating to phenomena which are entirely different, but which, analytically, present the most complete analogy —that of the uniform propagation of heat in a homogeneous medium, and that of the steady motion of an incompressible and frictionless liquid.

In short, the electrical problem is characterised by the existence of a function of the co-ordinates, which, vanishing at an infinite distance, has a constant value on each of the conductors, and for each point of the dielectric satisfies the ratio

$$\Delta V = 0,$$

the physical signification of which is very simple. X, Y and Z being the components of the force at a point P, the quantity $-\Delta V dv$ represents the total flow of force which proceeds from an element of volume dv taken at this point, and equation $\Delta V = 0$ expresses that this flow is nothing in the dielectric, or in the interior of a conductor; that is to say, where there is no electricity.

Let us now suppose that in a problem of statical electricity the insulating medium is replaced by a medium which conducts heat, and which is homogeneous, and *isotropic*; that is to say, which has the same properties in all directions; and let us suppose each of the electrified conductors replaced by sources which emit or which absorb heat, so as to maintain constant temperatures on the surfaces which are respectively equal in numerical value to the initial potentials, so that for each of the conductors $t = V$.

When once equilibrium is established, every point of the medium will be at a definite temperature, and *isothermal* surfaces can be traced; that is to say, surfaces of equal temperature, or of equal thermal level. It is clear that the temperature of a point P, comprised between two isothermal surfaces S and S', is independent of the situation of the sources, and that it will remain the same when these sources are suppressed, if the temperatures t and t' of these two surfaces are kept constant in any other way.

Fourier's hypothesis consists in assuming, what indeed may be regarded as the simple expression of facts, that heat travels from layer to layer; that the thermal effect of a point has no appreciable influence except on very near points; and that the hotter points

tend to raise the temperature of the colder ones. Fourier assumes, moreover, what is only true for a particular thermometric scale, that the interchanges of heat only depend on differences of temperature, and not on their absolute values.

The flow of heat which traverses an element dS of an isothermal surface S (Fig. 14) is, by symmetry, perpendicular to this surface, and to all the isothermal surfaces which it meets. The flow of heat dQ, which passes in unit time from the element dS to the infinitely near element dS′, is proportional to the surface dS$_1$, to the infinitely

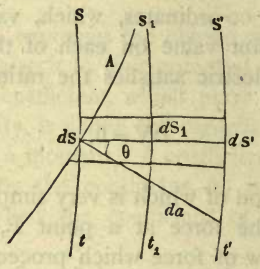

Fig. 14.

small difference of temperature $t-t'$, to a coefficient h which only depends on the nature of the medium; and, lastly, to a function of the distance of these elements.

We may therefore put

$$dQ = hdS\frac{t-t'}{\phi(e)}.$$

If we consider an intermediate temperature t_1, the flow of heat from dS to dS′ passes first through the element dS$_1$ at the distance e_1, which gives

$$dQ = hdS\frac{t-t_1}{\phi(e_1)} = hdS\frac{t-t'}{\phi(e)}.$$

Since the differences of temperature $t-t_1$ and $t-t'$ are, by continuity, proportional to the perpendicular distances, the function ϕ is proportional to the distance only. The flow of heat between two infinitely near corresponding elements, calling dt the variation of temperature measured in the direction of the flow of heat, and dn

the distance of the elements, may be expressed by the following formula :—

$$dQ = -kdS\frac{dt}{dn}.$$

The coefficient k is the *coefficient of conductivity of the medium.* It represents the flow of heat for unit surface between two parallel planes at unit distance, and whose temperatures differ by 1°.

In the present case, the value of the flow for unit surface at each point is

$$Q = -k\frac{dt}{dn};$$

it is proportional to the differential of the potential in reference to the perpendicular to the corresponding equipotential surface.

The flow of heat is the same across an element dA of any surface A bounded by the same tube. If θ is the angle which this element makes with the equipotential surface, and da the portion of the perpendicular to the element dA, comprised within the surfaces S and S′, we have

$$dQ = -kdA\cos\theta\frac{dt}{dn} = -kdA\frac{dt}{dn}.\frac{dn}{da} = -kdA\frac{\partial t}{\partial a}.$$

The flow of heat across an element of any given surface is therefore proportional to the partial differential of the temperature in the medium with respect to the perpendicular to this surface.

As equilibrium is established, the total flow of heat corresponding to any closed surface, not containing sources of heat, must be zero. Let u, v and w be the components of the flow at a point P, and $dxdydz$ an element of volume at the same point; the flow which enters by one of the surfaces $dydz$ is $udydz$, that which emerges by the opposite surface is equal to $\left(u+\frac{du}{dx}dx\right)dydz$, the difference is $dxdydz\frac{du}{dx}$; the total flow corresponding to the entire surface of the element is then

$$dxdydz\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right);$$

as this flow must be zero, it follows that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = -k\Delta t = 0.$$

It is evident, moreover, that the action of the system is not appreciable at great distances, and that the temperature t, which it determines, is zero at an infinite distance.

Hence, for every point of the medium and for the limits, the function t satisfies the same conditions as the function V. It is seen further that if the constant k is equal to unity, the numerical values of the flow of electrical force, and of the flow of heat during unit time, are identical in every point in the two problems.

71. Let us now consider the corresponding hydrodynamical problem. Let us imagine that the space originally occupied by the dielectric is filled by a frictionless and incompressible liquid; let us imagine, moreover, that the conductors are replaced by porous surfaces, so that the liquid has at each point of such surfaces, a normal velocity, equal to the original value of the electrical force at this point. The whole of the trajectories of the molecules which at the same moment have traversed the element dS of the surface of a conductor, form a liquid thread which issues perpendicularly, and yields the same supply in all sections. As there is nowhere any accumulation of liquid, the flow passing through a volume element $dxdydz$ taken at any point P, is equal to that which emerges; now if u, v, w, denote the components of the velocity at the point P, this condition is expressed by the equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

The motion is further inappreciable at an infinite distance; we thus see that the velocity at each point, depends on a function of the co-ordinates which satisfies the same conditions as the potential or the temperature. Lines of flow will coincide everywhere with the lines of force of the corresponding electrical problem, and at each point the electrical force and the velocity of the liquid will have the same numerical value.

The correlation which we have established is of great interest; for if it is clear that the analytical difficulties are exactly the same in the three kinds of problems, it is no less true that certain consequences present themselves more naturally in one order of ideas than in another, and it is clear that any result obtained in one case may be directly transferred, with its special interpretation, into another. We shall meet with many instances of this in the sequel.

72. ELECTRICAL CAPACITY.—We have designated *as the capacity of a conductor the charge which must be given to it to raise it to* unit *potential, when all the conductors which surround it are in communication with the earth.*

It follows, from this definition, that the capacity of a conductor depends not merely on its own shape, but on the shape and position of the conductors which surround it.

We shall represent this constant by the letter C. If, while the conditions remain the same, a charge M is imparted to the conductor, in virtue of the principle of superposition of conditions of equilibrium, its potential will be

$$V = \frac{M}{C};$$

from which follows

$$M = CV.$$

The problem of determining the capacity of a conductor in a given case, amounts to investigating the state of equilibrium of the system formed of the conductor in question, together with those surrounding it, these latter being in connection with the earth; it merges then into the general problem of equilibrium.

The word capacity has been borrowed by analogy from the theory of heat; but it is important to remark that while the calorific capacity of a body only depends on the nature and weight of the body, the electrical capacity of a conductor depends neither on its nature nor on its weight, but solely on its external shape and on the shape and position of all the adjacent conductors. The electrical capacity is not therefore a constant, fixed for the body in question, as is the thermal capacity.

73. SPHERE.—Let us consider a conducting sphere at a great distance from any other conductor. Let R be its radius, M its charge. By symmetry this charge forms a uniform layer on the surface; it satisfies, moreover, the condition of equilibrium, for its action on any internal point is null (42). The potential is, therefore, constant throughout the whole interior; its value at the centre is $\frac{M}{R}$; hence,

$$V = \frac{M}{R} \quad \text{or} \quad M = RV.$$

The capacity of the sphere is, therefore,

$$C = R;$$

it is equal to the radius. This example shows that the electrostatic capacity of a conductor is a linear quantity.

74. ELLIPSOID.—If a conductor bounded by the surface of an ellipsoid is covered by a homogeneous electrical layer, bounded itself by a second ellipsoidal surface concentrical and similarly placed to the former, the action of the layer on an internal point P is null.

Let us suppose, in fact, that this layer is very thin, and let us draw through the point P (Fig. 15) an infinitely slender cone $d\omega$, which cuts an element of surface dS at M at the distance u, and in the layer, a volume element the height of which along the radius

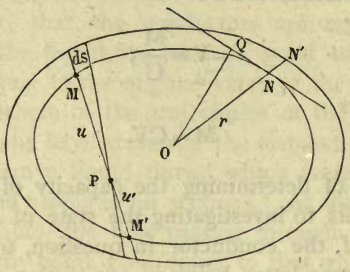

Fig. 15.

vector is du. The action at P of this volume element is in the direction of the radius vector, and calling ρ the density, its value is

$$\frac{\rho . u^2 d\omega . du}{u^2} = \rho d\omega du.$$

The action of the opposite element at M' is also $\rho d\omega du'$. As the heights du and du' are equal and the forces are directly opposed, their resultant is zero; this is also the case for all the elements of surface two by two, and the action of the entire layer on the point P is null. An electrical layer distributed on an ellipsoid according to this law will be then in equilibrium and will have a constant potential in the interior.

Let $(1+\alpha)$ be the ratio of similitude of the two surfaces supposed to be very close. The thickness of the layer at a point N is proportional to the distance of the tangent planes from the two homologous points N and N', and is equal to $p\alpha$, p denoting the perpendicular OQ let fall from the common centre on the tangent plane in N; the value of the surface density σ is

$$\sigma = \rho p \alpha.$$

The ellipsoid being represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the total mass of electricity is

$$M=\frac{4}{3}\pi abc\left[(1+\alpha)^3-1\right]\rho=4\pi abc\rho\alpha.$$

From this we get

$$\sigma=\frac{M}{4\pi abc}\cdot p=\frac{M}{4\pi abc}\cdot\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

The potential in the interior being constant, it is sufficient to calculate its value at the centre. We have then, if r be the radius ON,

$$V=\int\frac{\sigma dS}{r}=\frac{M}{4\pi abc}\cdot\int\frac{pdS}{r},$$

and the capacity of the ellipsoid is given by the equation

$$\frac{1}{C}=\frac{1}{4\pi abc}\int\frac{pdS}{r}.$$

75.—For an ellipsoid with three unequal axes the capacity is an elliptical function, but it may be easily obtained in the case of an ellipsoid of revolution.

Let us take as element of surface dS, the zone described by an element ds of the meridian curve, and let us suppose that the axis a is the axis of rotation. We have then

$$dS=2\pi yds=\frac{2\pi ydx}{\left(\frac{dx}{ds}\right)}=\frac{2\pi ydx}{\frac{py}{b^2}}=\frac{2\pi b^2}{p}dx,$$

from which

$$pdS=2\pi b^2dx.$$

This ratio shows already that the total charge of electricity is the same on all zones of equal height. On a very elongated ellipsoid, in the form of a double point, we may say that the linear density is constant.*

From this we get for the capacity of the ellipsoid

$$\frac{1}{C}=\frac{1}{4\pi abc}\int\frac{pdS}{r}=\frac{1}{2a}\int\frac{dx}{\sqrt{x^2+y^2}}.$$

If it is an ellipsoid of revolution about the major axis, e being the eccentricity, and C_1 the capacity, we have

$$\frac{1}{C^1}=\frac{1}{2a}\int_{-a}^{+a}\frac{dx}{\sqrt{b^2+e^2x^2}}=\frac{1}{2ae}\left\{l.\left[ex+\sqrt{b^2+e^2x^2}\right]\right\}_{-a}^{+a}=\frac{1}{2ae}\,l.\,\frac{1+e}{1-e}.$$

If the ellipsoid is one of revolution about the minor axis, the capacity C_2 is then

$$\frac{1}{C_2}=\frac{1}{2b}\int_{-b}^{+b}\frac{dy}{\sqrt{a^2-\frac{a^2e^2}{b^2}y^2}}=\frac{1}{2ae}\left[\text{arc sin}\,\frac{ey}{b}\right]_{-b}^{+b}=\frac{\text{arc sin}\,e}{ae}.$$

Each of these formulæ gives

$$C_1=C_2=a,$$

when the eccentricity is null; that is, when the ellipsoid becomes a sphere.

* This remark enables us to explain the *power of points*. On a very elongated ellipsoid in the form of a double point, it will be seen that the surface density is inversely as the diameter and constantly increases towards the extremity. If the insulating power of the air were itself without limit, the density and the tension, which is proportional to the square of the density, might increase without limit. But, as a matter of fact, when the tension has attained a certain value for a given pressure of air, the electricity passes from the conductor into the masses of air surrounding it, aud these being charged with electricity escape along the lines of force, producing the phenomena known as the electrical *aura* and of the *brush discharge*.

76.—An elliptical plate may be regarded as a very flattened ellipsoid. We have then, at the limit,

$$\lim.\left(\frac{p}{c}\right)=\frac{1}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}.$$

The density on the plate is expressed by

$$\sigma=\frac{M}{4\pi ab}\,\frac{1}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}},$$

and it will be seen that the lines of equal density are concentric and similarly placed ellipses.

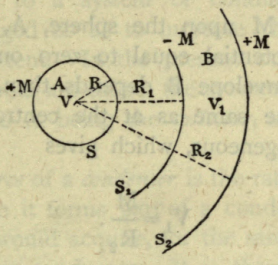

Fig. 16.

For a circular plate, we have simply, at a distance r from the centre,

$$\sigma=\frac{M}{4\pi a^2}\cdot\frac{1}{\sqrt{1-\frac{r^2}{a^2}}}=\frac{M}{4\pi a\sqrt{a^2-r^2}},$$

and the capacity of the plate is reduced to

$$C=\frac{2a}{\pi}=\frac{a}{1,571}.$$

77. CONCENTRIC SPHERES.—Let us suppose that a conducting sphere is surrounded by a conductor, bounded by two spherical surfaces concentric to the first.

Let R be the radius of the sphere A (Fig. 16), R_1 and R_2 those of the concentric envelope B, which at first we will suppose is insulated. If an electric charge M is given to the sphere A, the envelope B acquires (58) a charge equal to $-M$ upon its inner surface S_1, and a charge $+M$ upon its outer surface S_2. The value of the potential at the centre of the sphere is obviously

$$V=\frac{M}{R}-\frac{M}{R_1}+\frac{M}{R_2}=M\left[\frac{1}{R}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right].$$

The capacity C of the inner sphere being the charge which corresponds to $V=1$, we have

$$\frac{1}{C}=\frac{1}{R}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

The two layers $+M$ upon the sphere, A, and $-M$ upon the surface S_1, have a potential equal to zero on the outside. The potential V_1 of the envelope B depends, then, solely on the outer layer $+M$, and is the same as at the centre of the sphere S_2, supposed to be homogeneous, which gives

$$V_1=\frac{M}{R_2}.$$

If the envelope B is connected with the earth, its potential becomes zero, the external charge $+M$ disappears, and the capacity of the sphere is then

$$\frac{1}{C}=\frac{1}{R}-\frac{1}{R_1}=\frac{e}{RR_1},$$

e being the thickness of the dielectric.

If the thickness of the dielectric is small in comparison with the radius of the sphere, we may neglect the difference between R and R_1, and take $\frac{R^2}{e}$ instead of $\frac{RR_1}{e}$ for the capacity. If this capacity be expressed as a function of the surface of the sphere, we have

$$C=\frac{R_2}{e}=\frac{4\pi R_2}{4\pi e}=\frac{S}{4\pi e}.$$

The charge required to raise the sphere to potential V is expressed by

$$M = \frac{SV}{4\pi e}.$$

It will be seen that in all cases the concentric shell, by increasing the capacity of the sphere, has the effect of diminishing the potential relative to a given charge, or conversely, of increasing the charge relative to a given potential.

78. CONDENSERS. — The presence of the envelope makes it possible to accumulate or condense on the sphere A, a greater quantity of electricity, for the same potential, than if this envelope did not exist. The same effect would be produced upon any conductor A, by the proximity of a second conductor B in connection with the earth, or insulated, but with a charge null, for this conductor diminishes the value of the potential for a given charge. The term *condenser* is applied to a system of conductors separated by a dielectric, and arranged so as to increase the capacity of one of them to a notable extent. In the present case, the sphere and its envelope constitute what are called the *armatures*, or *coatings* of the condenser, the sphere A being the *collector*, and the sphere B the *condenser*.

The *condensing force* of a *condenser* is the ratio between the charge of the collector when it forms part of a condensing apparatus, and the charge which it would acquire, for the same potential, if it were distant from any other conductor. It is therefore the ratio of the capacities of the collector in these two circumstances. In the spherical condenser with concentric surfaces, the outer coating of which is in connection with the earth, the value of the condensing force is

$$\frac{C}{R} = \frac{R}{e}.$$

The application of the condensing force presents no interest; the only magnitude which requires to be known is the capacity of a condenser.

79. LEYDEN JAR.—A *Leyden jar* is a glass vessel coated outside and inside with metal foil, with the exception of a part near the opening, so that the coatings may not communicate. A conducting rod passing through the neck is connected with the internal coating.

The system of these two conducting surfaces constitutes an almost closed condenser.

The preceding problem corresponds to the case of a spherical jar of constant thickness; the influence of the small zone, which must be cut off from the outer surface, to allow of communication with the interior, may be neglected; the capacity is therefore represented by $\frac{S}{4\pi e}$, and the charge by the formula

$$M = \frac{SV}{4\pi e}.$$

It is easily seen that this formula applies equally to a Leyden jar of any form, of constant and very small thickness, the coatings of which cover the whole surface outside and inside.

Let S_1 and S_2 (Fig. 17) be the two opposite surfaces of the two coatings, V_1 and V_2 their potentials. As the outer coating completely surrounds the inner one, the electrical masses on these two surfaces

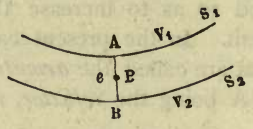

Fig. 17.

are equal, and of opposite signs (57). The value of the force for a point P of the dielectric, σ being the density at A of the inner layer, is

$$F = -\frac{dV}{dn} = 4\pi\sigma.$$

As the thickness e is supposed to be very small, the differential $\frac{dV}{dn}$ is virtually equal to $\frac{V_2 - V_1}{e}$, which gives

$$e = \frac{1}{4\pi} \cdot \frac{V_1 - V_2}{e}.$$

The charge of a surface element dS is σdS, and the total charge is

$$M = \int \sigma dS = \frac{V_1 - V_2}{4\pi} \int \frac{dS}{e}.$$

If the thickness is constant, then, putting $V = V_1 - V_2$, we have

$$M = \frac{V_1 - V_2}{4\pi} \cdot \frac{S}{e} = \frac{VS}{4\pi e}.$$

The capacity of the jar—in other words, the charge which corresponds to a difference of unit potential between the two coatings—is expressed in the same way as for spherical condensers; that is to say,

$$C = \frac{S}{4\pi e}.$$

The charge depends thus only on the difference of potentials, and not at all on their absolute values: this result could be readily foreseen, as the force itself only depends on this difference.

80. Concentric Cylinders.—Given two concentric cylinders with circular bases of the radii R_1 and R_2, at the potentials V_1 and V_2. Consider the very small angle $d\omega$ formed by two planes passing

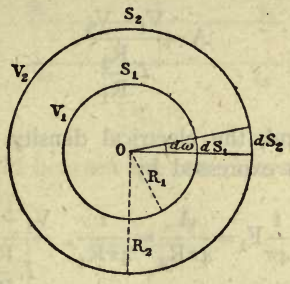

Fig. 18.

through the axis, and cut this by two planes perpendicular to the axis, one of which we shall take for the plane of the figure (Fig. 18). All the lines of force being by symmetry perpendicular to the common axis, the volume thus determined is a tube of force; the surfaces which it intercepts on the two cylinders are to each other as the arcs dS_1 and dS_2. If, then, F_1 and F_2 are the values of the force at the distances R_1 and R_2, we have

$$F_1 dS_1 = F_2 dS_2.$$

As, moreover,

$$d\omega = \frac{dS_1}{R_1} = \frac{dS_2}{R_2},$$

we obtain, by substitution,

$$F_1 R_1 d\omega = F_2 R_2 d\omega, \quad \text{whence} \quad F_1 R_1 = F_2 R_2 = A.$$

The force, therefore, at any point of the dielectric is inversely as the distance from the axis.

If F be the force, and V the potential at any distance R, we have

$$F = \frac{A}{R} = -\frac{dV}{dR},$$

and therefore,

$$V = -A\int\frac{dR}{R}.$$

Extending this integral to the volume comprised between the surfaces S_1 and S_2, we have

$$V_2 - V_1 = -Al.R_2 + Al.R_1 = -Al.\frac{R_2}{R_1}.$$

From which is deduced for the constant A,

$$A = \frac{V_1 - V_2}{l.\dfrac{R_2}{R_1}}.$$

On the other hand, the electrical density σ, at the surface of the inner cylinder, is expressed by

$$\sigma = \frac{1}{4\pi}F_1 = \frac{A}{4\pi R_1} = \frac{1}{4\pi R_1}.\ \frac{V_1 - V_2}{l.\dfrac{R_2}{R_1}}.$$

Let S be the extent of surface comprised between the two planes perpendicular to the axis; the mass M distributed on this surface is

$$M = S\sigma = \frac{(V_1 - V_2)S}{4\pi R_1 l.\dfrac{R_2}{R_1}}.$$

If L is the length of the cylinder, thus determined,

$$S = 2\pi R_1 L,$$

which gives, finally,

$$M = \frac{L}{2l.\dfrac{R_2}{R_1}}(V_1 - V_2).$$

The capacity of a cylindrical condenser for unit length is then

$$C = \frac{1}{2l.\left(\frac{R_2}{R_1}\right)}.$$

This is a problem of great practical importance, as it represents the case of telegraph cables, which are made of conducting wires surrounded by an insulator, which in turn is itself protected from injury by a metallic coating.

81. PLANE CONDENSERS.—Consider two conductors bounded by plane parallel surfaces S_1 and S_2, at the distance e, and at the potentials V_1 and V_2 (Fig. 19). At a distance from the edges, which is very great compared with the thickness of the dielectric, the lines of force are parallel straight lines perpendicular to the surface in question.

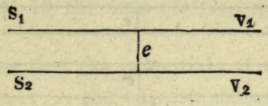

Fig. 19.

The electrical field between the two planes is then uniform; the force is expressed by

$$F = \frac{V_1 - V_2}{e},$$

and the density on the surface S is

$$\sigma = \frac{F}{4\pi} = \frac{V_1 - V_2}{4\pi e}.$$

The electrostatic pressure, that is to say the force exerted upon unit surface, is

$$p = 2\pi\sigma^2 = \frac{1}{8\pi}\left(\frac{V_1 - V_2}{e}\right)^2;$$

it is proportional to the square of the difference of potential, and inversely as the square of the distance.

Suppose that a portion a, of the surface S, at a great distance from the edges, is movable, and that, being in contact with the general surface, so as to have the same potential, it is maintained

there by an antagonistic force; this surface will have a uniform electrical layer, and the force P, necessary to resist the electrical attraction, is expressed by

$$P = pa = \frac{a}{8\pi} \cdot \left(\frac{V_1 - V_2}{e}\right)^2.$$

Sir W. Thomson has made use of this property in the construction of his absolute electrometers; that portion of the surface S which surrounds the movable part to keep it at a constant density is called the *guard-plate.*

82. Let us now suppose that a conducting plate A, at potential V_1 (Fig. 20), is placed between two conductors B and B', terminated by surfaces parallel to those of the plate, one at the distance e, and potential V_2, the other at the distance e' and potential V'_2.

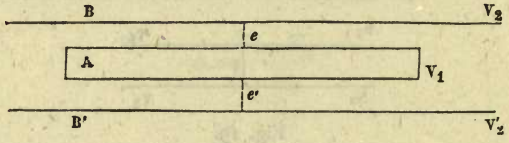

Fig. 20.

The density on the conductor A, at a great distance from the edges, is $\frac{V_1 - V_2}{4\pi e}$ for the upper face, and $\frac{V_1 - V'_2}{4\pi e'}$ for the lower face, so that the charge which corresponds to unit surface of the plate is

$$\frac{1}{4\pi}\left(\frac{V_1 - V_2}{e} + \frac{V_1 - V'_2}{e'}\right).$$

Disregarding the variation of density at the edges, the total charge of the plate, of surface S, is thus

$$M = \frac{S}{4\pi}\left(\frac{V_1 - V_2}{e} + \frac{V_1 - V'_2}{e'}\right).$$

If the potentials V_2 and V'_2 are equal, we have simply

$$M = \frac{(V_1 - V_2)\,S}{4\pi}\left(\frac{1}{e} + \frac{1}{e'}\right),$$

so that the capacity of this condenser is

$$C = \frac{S}{4\pi}\left(\frac{1}{e} + \frac{1}{e'}\right),$$

an expression which agrees with that already obtained (79) for closed conductors.

83. CAPACITY OF A SYSTEM OF CONDUCTORS.—Let us consider a number of different conductors whose electrical capacities are respectively C, C', C'',..... and which are so arranged that their inductive action on each other is zero. If all these conductors, being at the same potential, are joined by means of conductors whose capacity may be neglected, fine wires for instance, no exchange of electricity will take place, for they were all at the same potential, and this potential will not change.

They form thus a single conductor, the charge of which is equal to the sum of the original charges. The electrical capacity of the system is equal to the sum of the capacities of the separate conductors.

Let us now suppose that the potentials of the original conductors are different V, V', V'',..... the corresponding charges are

$$M = VC, \qquad M' = V'C', \qquad M'' = V''C'' \ldots\ldots$$

All these charges being regularly distributed upon the single conductor formed by the system, will produce a potential V_1 given by the equation

$$V_1 C_1 = VC + V'C' + V''C'' + \ldots\ldots,$$

whence

$$V_1 = \frac{VC + V'C' + V''C'' + \ldots\ldots}{C + C' + C'' + \ldots\ldots}.$$

This expression is frequently used in experimental researches. It will be remarked how analogous it is to that which represents the temperature resulting from the mixture of several different bodies at different temperatures.

84. BATTERIES.—This term is applied to the system formed of several Leyden jars, or condensers of any kind, which are connected with each other.

If the condensers are virtually closed, as is the case with ordinary Leyden jars, the external action of each of them is insignificant, and they can be brought near each other without exerting any appreciable influence.

Connection may be made in two ways:—

1st. All the inner coatings may be connected with each other, on the one hand, and all the outer coatings on the other; the battery is then said to be arranged for *quantity*. The whole forms a condenser whose capacity is equal to that of the capacities of all the jars separately. If the battery contains p identical jars, each with the capacity C, the capacity C_1 of the battery is

$$C_1 = pC.$$

2nd. All the jars being insulated, the outer coating of one is connected with the inner coating of the following one; the inner coating of the first jar is charged to potential V_1, the outer coating of the last jar being at potential V_2, and all the intermediate coatings being insulated; this arrangement is said to be in *series* or *cascade*.

85. CHARGE BY CASCADE.—The first jar receives a charge m on its inner coating, and assumes the potential V_1; an equal and opposite charge $-m$ is produced on the surface of the dielectric next the outer coating. The conductor formed of this coating and the inner coating of the second jar being insulated, will take a charge $+m$, which is distributed regularly upon this conductor, as if the internal charges did not exist, and produces there a potential V'. The greater part of this charge passes to the inner coating of the second jar, the capacity of which is very great in reference to that of the conductor in question. Continuing this reasoning, it will be seen that the inner coatings have continually decreasing charges, but the diminution is very small, and we may consider all the jars as having the same internal charge $+m$, the successive potentials being V_1, V', V'', ... V_2.

If the battery contains p jars, we may put

$$m = C(V_1 - V') = C'(V' - V'') = \ldots\ldots = C^{(p-1)}(V^{(p-1)} - V_2),$$

whence

$$\frac{m}{C} = V_1 - V',$$

$$\frac{m}{C'} = V' - V'',$$

$$\vdots \qquad \vdots$$

$$\frac{m}{C^{p-1)}} = V^{(p-1)} - V_2.$$

Adding all these equations, we get

$$m\left(\frac{1}{C}+\frac{1}{C'}+\frac{1}{C''}\cdots\cdots\right)=V_1-V_2.$$

Hence the charge of the first jar, which is the only one that receives electricity directly, is

$$m=\frac{V_1-V_2}{\frac{1}{C}+\frac{1}{C'}+\frac{1}{C''}\cdots\cdots}.$$

We have thus, for the capacity C_1 of the battery,

$$C_1=\frac{1}{\frac{1}{C}+\frac{1}{C'}+\cdots\cdots},$$

$$\frac{1}{C_1}=\frac{1}{C}+\frac{1}{C'}+\frac{1}{C''}=\cdots\cdots$$

If the jars are identical, the capacity of the battery has become p times less than that of each of the jars.

This arrangement may appear unfavourable, since its effect is to diminish greatly the capacity of the battery; yet it presents great advantages for certain experiments. Leyden jars can only sustain a limited difference of potential, beyond which their coatings discharge themselves along the surface of the glass, and even sometimes through the mass of the glass itself, which is then traversed by a spark. By means of a battery in cascade, the total difference of potential may be distributed in stages on the successive jars.

This, for instance, is the arrangement adopted in the ordinary Holtz machines, where the capacity of the conductors is increased by connecting each of them with the inner coating of a Leyden jar; care however is taken to join these in cascade, so as to maintain the maximum difference of potential, and therefore the greatest striking distance which the play of the machine allows.

When a large number of jars are available, they may be joined together for quantity so as to form several batteries, which in turn are arranged in cascade. In this way the whole of the potential which a machine can yield may be utilised, and the maximum of effect obtained with the least expenditure of electricity.

86. General Problem of the Reciprocal Influence of Two Insulated Conductors.—Murphy's Method.—In order to determine the distribution of electricity on two insulated conductors A and B, charged with the total masses M_a and M_b and only submitted to their reciprocal action, it is sufficient if we know for each of them:

1st The capacity and the distribution on the surface when it is insulated and not subject to any external induction;

2nd. The distribution of the electricity induced on the surface when it is in connection with the earth, and is subject to the inductive action of an electrical mass placed at any point outside it.

Let m be the capacity of the conductor A alone—that is to say, the charge which would then produce potential *unit*.

Let us fix this mass, the distribution of which is known, and let us place in the desired position the conductor B in connection with the earth. This will be at potential zero, and will become charged with a known mass of the opposite electricity $-m'$.

In like manner let us fix the mass $-m'$ on B. Let this conductor be insulated, and let the first one be connected with the earth; this latter will acquire a mass m_1 at potential zero.

In like manner let the mass $+m_1$ be fixed on A, an induced layer $-m''$ will be obtained on B, and so forth.

Continuing in the same manner, we shall successively obtain the masses $m, m_1, m_2 \ldots$ on the former, and $-m', -m'', -m''' \ldots$ on the latter, each of them tending to verge rapidly towards zero.

The superposition of all the layers $m, m_1, m_2 \ldots$ on A, and of all the layers $-m', -m'', -m'''$ on B will result in a state of equilibrium with zero potential on B, and potential equal to unity on A. In fact, the successive layers m and $-m'$, m_1 and $-m'', \ldots$ taken in pairs, give zero potential on B; the layers $-m'$ and m_1, $-m''$ and $m_2, \ldots$ give, in like manner, zero potential on A. We have only thus to consider the mass m on the first conductor, which produces a potential equal to unity.

Putting

$$C_a = m + m_1 + m_2 + \ldots\ldots,$$
$$C'_a = m' + m'' + m''' + \ldots\ldots,$$

we see that C_a represents the capacity of the insulated conductor A in the presence of the conductor B connected with the earth, and $-C'_a$ the coefficient of electricity induced on B (69).

Multiply these two masses by V_a, the respective charges C_aV_a and $-C'_aV'_a$ correspond to a state of equilibrium with zero potential on B, and potential equal to V_a on A.

Reversing the functions of the conductors, we shall obtain the masses $C_b V_b$ on B and $-C'_b V'_b$ on A, corresponding to a new state of equilibrium, with zero potential on A, and potential V_b on B.

The superposition of these two states of equilibrium gives a new state of equilibrium with the addition of the potentials on each of the conductors—that is to say, the potential V_a on A and V_b on B. The total charges of the two conductors are these

$$M_a = C_a V_a - C'_b V_b.$$
$$M_b = C_b V_b - C'_a V_a.$$

These equations enable us to calculate the total masses of the two conductors when the potentials are known.

In like manner the potentials may be deduced as functions of the masses, which gives

$$V_a = \frac{C_b M_a + C'_b M_b}{C_a C_b - C'_a C'_b},$$
$$V_b = \frac{C_a M_b + C'_a M_a}{C_a C_b - C'_a C'_a}.$$

87. Reciprocal Action of Two Electrified Conductors.—The preceding method enables us to determine the distribution of electricity on the two conductors, for the final density at each point is the sum of the densities relative to the various superposed layers, and by hypothesis we know the law of distribution for each. We have then all the elements needed for calculating the action exerted between the two bodies; the problem only presents then difficulties of calculation.

This force consists of the action of each of the two layers $C_a V_a$ and $-C'_b V_b$ of the body A, on the two layers $C_b V_b$ and $-C'_a V_a$ of the body B. The potentials being supposed positive, the action of $C_a V_a$ is made up of two terms—one repulsive, proportional to the product $V_a V_b$ of the two potentials, and the other attractive proportional to V_a^2.

The action of $-C'_b V_b$ comprises also two terms, one attractive proportional to V_b^2 and the other repulsive proportional to the product $V_a V_b$.

Calling a, b, and c coefficients which depend on the form of the body and on their distance, the reciprocal action R, considered as repulsive, has an expression of the form

$$R = 2cV_a V_b - aV_a^2 - bV_b^2.$$

If the conductors A and B are identical, and arranged symmetrically, the coefficients a and b are equal, and the formula becomes

$$R = 2cV_aV_b - a(V_a^2 + V_b^2).$$

We have assumed that the action of the two bodies reduces to a single resultant. If it were not so, the same reasoning would apply to the two resultants by which the whole of the forces may be replaced.

As a matter of fact the calculations required by this method for determining the coefficients C_a, C_b, C'_a and C'_b, and the resultant R, are extremely tedious even in the simplest cases. We shall afterwards explain the application which Sir W. Thomson has made of it to calculating the reciprocal influence of two spheres.

CHAPTER V.

WORK OF ELECTRICAL FORCES.

88. Electrical Energy.—When different electrical conductors are connected with the earth, the system reverts to the neutral state, and in doing so performs work which is necessarily positive. Any given system of electrical conductors possesses a store of available energy corresponding to this work; it is a *potential* energy, which we may simply speak of as *electrical energy*.

The electrification of a system, requires the expenditure of an amount of work equal to the potential energy which it possesses in this new condition.

When two conductors are connected, a change is in general produced in the distribution of the electrical masses, and this modification corresponds to a positive work. The electrical energy of a system of conductors is therefore equal, or superior, to that of the system obtained by connecting all these conductors in any way whatever.

When the system contains an electrified insulating body, we may look upon the several electrified masses, with which the body is charged, as belonging to infinitely small conductors. If all the masses are connected together, the energy diminishes. The energy of a system of bodies, each of which possesses a given mass, is therefore a minimum when all the bodies are conductors.

The potential energy of a system may be measured either by the work expended in electrifying it, or by the work which is performed by its discharge.

89. Energy of a Single Conductor.—Let us first consider a single conductor of capacity C, and let us suppose that a charge M has been given to it, which raises it to potential V. To increase the charge by dM, this quantity dM of electricity must be brought from infinity, or from the earth, to the conductor, and the work expended in this operation is equal to $V dM$.

The increase dW of the energy of the conductor is therefore

$$dW = V dM = \frac{M dM}{C}.$$

When the mass of electricity changes from M_0 to M_1, the increase of energy is

$$W_1 - W_0 = \frac{1}{2C}(M_1^2 - M_0^2).$$

As the energy vanishes with the mass, we see that the energy which corresponds to the mass M is

$$W = \frac{M^2}{2C} = \frac{1}{2} CV^2 = \frac{1}{2} MV.$$

Thus *the electrical energy of a single conductor is proportional to the square of the charge, or to the square of the potential.*

90. ENERGY OF A SYSTEM OF CONDUCTORS.—Let there be any number of conductors A_1, A_2, A_3, ... having charges M_1, M_2, M_3, ... with the potentials V_1, V_2, V_3, ...

If the density of each point is multiplied by x, a new state of equilibrium is obtained, in which the potentials are multiplied by the same factor x. There is the charge xM_1 on A_1 at the potential xV_1, xM_2 on A_2 at the potential xV_2, etc.

If we increase x by dx, the masses and the potentials are multiplied by $x+dx$, and the increase of charge in the conductor A_1 is $M_1 dx$. The corresponding work lies between $M_1 dx . xV_1$ and $M_1 dx(x+dx)V_1$; it is therefore, within an infinitely small expression of the second order, equal to $M_1 V_1 x dx$. This is also the case with the other conductors, so that the variation of energy of the system is

$$dW = (M_1V_1 + M_2V_2 + \ldots\ldots) x dx = x dx \sum MV.$$

Between the two values x_0 and x_1 the increase of energy is

$$W_1 - W_0 = \frac{x_1^2 - x_0^2}{2} \sum MV.$$

If we make $x_0 = 0$ and $x_1 = 1$, which amounts to supposing that

to reach the state in question we started from the neutral state, we have simply

$$W=\frac{1}{2}(M_1V_1+M_2V_2+\dots..)=\frac{1}{2}\Sigma MV.$$

We thus see that *the energy of a system of conductors is equal to the half-sum of the products of each mass by the corresponding potential.*

91. A conductor which remains insulated during the charge is merely electrified by induction, and its total charge is zero; there is no term, therefore, in the sum of the products, which corresponds to an insulated conductor.

In like manner, a conductor kept in connection with the earth remains at zero potential, and does not enter into the expression for the energy.

It must however be remarked that these two kinds of conductors affect the value of the energy, by modifying the influence of the capacities, and therefore the potentials, of the electrified bodies.

Lastly, the same formula holds for the case of insulating bodies, however electrified. Each of the volume elements of an insulating body may, in fact, be considered as an infinitely small conductor on which the corresponding electrical mass is distributed. In this case the preceding sum becomes an integral; calling ρ the electrical density, and V the potential on the volume element dv, the energy of the system is expressed by

$$W=\frac{1}{2}\int Vd\mathrm{M}=\frac{1}{2}\int V\rho dv.$$

The energy accumulated by electrification on a system of conductors is expended when the system is discharged, and may be transformed into mechanical work, or into an equivalent effect: disengagement of heat, chemical action, etc.

92. If electricity were a material substance, the masses constituting the electrical layers would acquire a certain *vis viva* during the discharge, in virtue of which they would, like a pendulum, pass beyond their position of equilibrium, so as to restore to the system a fraction of its initial energy; a succession of discharges alternately in opposite directions would be produced, until the heat disengaged upon the conductors had exhausted the whole of the available energy, and final equilibrium would only be reached after a certain number of oscillations. Experiment shows, indeed, that under

certain conditions the discharges have a distinctly oscillatory character; but we shall see that these oscillations may be explained in a totally different manner. Hence no conclusion can be drawn in favour of the hypothesis which assigns a certain inertia to electrical masses, and in the present state of science no decisive fact can be claimed for or against this hypothesis.

93. DISCHARGE OF BATTERIES.—QUANTITY BATTERY.—*Total Discharge.*—We have seen that the capacity C_1 of a battery arranged for quantity is equal to the sum of the capacities of each of the jars.

If the total energy of the battery is transformed with heat during the discharge, then, calling J the mechanical equivalent of the unit of heat, and Q the heat disengaged, we have

$$W = \frac{1}{2}MV = \frac{1}{2}\frac{M^2}{C_1} = C_1V^2 = JQ.$$

If the battery consists of p identical jars, of capacity C, the formula becomes

$$W = JQ = \frac{1}{2p}.\frac{M^2}{C} = \frac{1}{2}pCV^2.$$

We thus see that, for a given charge, the energy, or the heat disengaged, is inversely as the number of jars, and that for a given potential the energy is proportional to the number of jars.

94. *Incomplete Discharge.*—Let us consider two batteries of the capacities C_1 and C_2, the former charged with a mass M and the second in the neutral state, the outer coatings being connected with the earth. Let us suppose that instead of discharging the first, we join the coatings so as to form a single jar of the capacity $C_1 + C_2$. The discharge is said to be *incomplete;* it represents a loss of energy, and produces a disengagement of heat. Before contact, the potential energy of the first battery was

$$W_1 = \frac{1}{2}\frac{M^2}{C_1}.$$

After contact, the energy of the system has become

$$W_2 = \frac{1}{2}\frac{M^2}{C_1 + C_2}.$$

The energy expended in the discharge is then

$$W_1 - W_2 = \frac{M^2}{2}\left(\frac{1}{C_1} - \frac{1}{C_1 + C_2}\right) = \frac{M^2}{2C_1} \cdot \frac{C_2}{C_1 + C_2}.$$

The proportion of the initial energy which has been expended is

$$\frac{W_1 - W_2}{W_1} = \frac{C_2}{C_1 + C_2} = \frac{1}{1 + \frac{C_1}{C_2}}.$$

Let us suppose that the first battery consists of p_1 jars of the surface S_1 and thickness e_1, and the second of p_2 jars of the surface S_2 and the thickness e_2, we shall have

$$\frac{C_1}{C_2} = \frac{p_1}{p_2} \cdot \frac{S_1}{S_2} \cdot \frac{e_2}{e_1},$$

which gives

$$\frac{W_1 - W_2}{W_1} = \frac{1}{1 + \frac{p_1}{p_2} \cdot \frac{S_1}{S_2} \cdot \frac{e_2}{e_1}}.$$

95. DISCHARGE OF A BATTERY IN CASCADE.—The capacity C_1 of a battery arranged in cascade is connected with the capacities C, C′, C″... of the several insulated jars by the expression (85)

$$\frac{1}{C_1} = \frac{1}{C} + \frac{1}{C'} + \frac{1}{C''} + \ldots\ldots = \Sigma\frac{1}{C}.$$

The expression for the potential energy of the system only comprises the term relative to the first jar; for all the other conductors are insulated, or in connection with the earth, while being charged. We have thus

$$W = \frac{1}{2}\frac{M^2}{C_1} = \frac{1}{2}C_1V^2.$$

If the battery consist of p identical jars,

$$C_1 = \frac{C}{p},$$

which gives for the energy

$$W = \frac{1}{2}p\frac{M^2}{C} = \frac{1}{2}\frac{CV^2}{p}.$$

For a given charge, the energy of the cascade would be greater

than that of a single jar; but for a given potential it would be p times less. It is the exact opposite of charge by quantity.

All the laws relating to the discharge of batteries have been experimentally established by M. Riess.

On the whole, then, in working at a constant potential—that is to say, with a constant source of electricity, the best combination that can be made with a given number of jars, so as to obtain the maximum energy in the discharge, is to join them in quantity, provided always that the jars can sustain the maximum potential of the source. If, on the other hand, only a limited supply of electricity is available, it is best to arrange them in cascade.

The first is the case most frequently met with in electrical machines; but as they usually produce very high potentials, it is often advantageous to select a suitable combination of the jars by which these high potentials may be used and at the same time the charge be economized.

96. Electrical Work in the Displacement of Insulated Conductors.—*Conductors with a Constant Charge.*—The value of the potential energy of a system of conductors is

$$W = \frac{1}{2}\sum MV.$$

When the relative position of these conductors is changed, without in any way connecting them, a positive or negative work of the electrical forces is, in general, produced, and therefore the energy of the system is altered. If the conductors are left to themselves they obey the electrical actions which urge them; the work of these forces is positive and corresponds to a loss of energy in the system. If, by any external work, the system experiences a deformation in a direction contrary to that of the electrical actions, the energy increases to a corresponding extent.

Hence, calling $d'T$ the work of the electrical forces, and dW the corresponding variation of energy, we have at each moment

$$(1) \qquad dW + d'T = 0.$$

The energy of conductors which are left to their reciprocal actions tends therefore towards a minimum.

We have, moreover, the general expression,

$$dW = \frac{1}{2}\sum M dV + \frac{1}{2}\sum V dM;$$

but in the present case the last term is zero, for the charge is constant on each of the conductors; there simply remains

$$dW = -\frac{1}{2}\sum' M dV.$$

A conductor originally in the neutral state would be drawn into the electrical field. Hence the effect of the presence of this conductor is to diminish the energy of the system.

97. *Conductors at Constant Potential.*—Let us now consider the case of conductors kept at constant potentials by sources of electricity placed outside the field of action.

We shall suppose that the various conductors A_1, A_2, $A_3 \ldots$, charged with quantities M_1, M_2, $M_3 \ldots$, and to the potentials V_1, V_2, $V_3 \ldots$, communicate separately with bodies of the capacities C_1, C_2, $C_3 \ldots$, withdrawn from any external influence—for instance, closed condensers the outer coating of which is connected with the earth. This case comes under that which we have been considering; if W_a is the energy of the conductors and W_c that of the condensers, the energy of the system is

$$W = W_a + W_c.$$

If the system undergoes any deformation without the intervention of extraneous energy, the theorem (1) applies and gives

$$dT + dW_a + dW_c = 0. \tag{2}$$

The energy of the conductor is

$$W_a = -\frac{1}{2}\sum' MV,$$

and therefore

$$dW_a = -\frac{1}{2}\sum' M dV + -\frac{1}{2}\sum' V dM.$$

For the energy of the condensers, the capacity of which is unchanged, we shall take the expression

$$W_c = -\frac{1}{2}\sum' CV^2,$$

from which is deduced

$$dW_c = \Sigma CV dV.$$

Lastly, the total charge $M + CV$ for any system consisting of a conductor and the corresponding condenser is constant; we have then

$$dM + C dV = 0,$$

and therefore

$$V dM + CV dV = 0;$$

which gives for the conductor and condenser together

$$\Sigma V dM + \Sigma CV dV = 0.$$

Taking into account this latter relation, equation (2) may be written

$$dW_a = \frac{1}{2}\Sigma M dV - \frac{1}{2}\Sigma CV dV = \frac{1}{2}\Sigma M dV - \frac{1}{2}dW_c.$$

We have then

$$(3) \qquad 2dW_a + dW_c = \Sigma M dV = -\Sigma \frac{M dM}{C}.$$

This equation holds, whatever be the capacities of the condenser. There is nothing to prevent our considering the capacities as being infinitely large in reference to those of the conductors, so that the variations of potential dV_1, dV_2.. and the variations of energy $M_1 dV_1$, $M_2 dV_2$... are absolutely negligable. We come then to the case of conductors kept at constant potentials by external sources, and equation (3) reduces to

$$2dW_a + dW_c = 0,$$

whence

$$-(dW_a + dW_c) = dW_a;$$

which gives finally, from equation (1),

$$dT = dW_a.$$

Thus, when conductors are kept respectively at constant potentials, the energy of the system, for a given deformation, increases by a quantity equal to the work of the electrical forces. This work is positive if the system is left to itself; like the corresponding

increase in the energy, it is borrowed from the sources which keep the potentials constant. The sources yield then, at every moment, a quantity of energy which is divided into two equal parts; one serves to perform the work dT of the electrical forces, the other goes to increase by dW_a the electrical energy of the system.

In this case the energy of the system tends towards a maximum.

98. We shall proceed to apply these theorems to the following problem, which may serve as basis of the theory of symmetrical electrometers.

Let us suppose that a system of conductors is formed of two fixed unlimited cylinders A and B (Fig. 21) having a common axis,

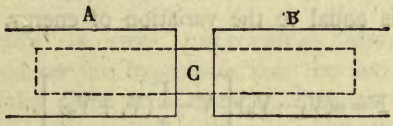

Fig. 21.

and of a cylinder concentric with the preceding ones, movable along this axis, the length of the inner cylinder C being, moreover, so great that the density at each end only depends on that of the nearest fixed conductor. Let V_1, V_2, and V be the potentials of these three bodies, and A_0, B_0, and C_0 the charges which they possess when the movable cylinder is in a position symmetrical with the two others.

If the cylinder C is displaced by a small quantity x, towards the right for instance, the distribution of electricity on the various surfaces near the opening and at the ends is not modified; we have merely on this side increased, by a quantity proportional to x, the surface on which the electrical density is uniform and proportional to the difference of potentials of the adjacent conductors The right half of the movable cylinder will have gained a quantity of electricity proportional to x, and the fixed cylinder B an equal quantity of the contrary electricity; the opposite effect will be produced on the other side.

Thus, calling A_0, B_0, C_0 the initial charges on the three cylinders, A, B, and C the new charges, and α the capacity for unit length of the inner cylinder at some distance from the middle and from the ends,

$$C = C_0 + \alpha x (V - V_2) - \alpha x (V - V_1) = C_0 + \alpha x (V_1 - V_2),$$
$$B = B_0 - \alpha x (V - V_2),$$
$$A = A_0 + \alpha x (V - V_1).$$

The variation of energy is then

$$W - W_0 = \frac{1}{2}\alpha x\left[(V_1 - V_2)V - (V - V_2)V_2 + (V - V_1)V_1\right]$$

$$= \alpha x\left[V(V_1 - V_2) - \frac{1}{2}\left(V_1^2 - V_2^2\right)\right]$$

$$= \alpha x\,(V_1 - V_2)\left[V - \frac{1}{2}(V_1 + V_2)\right].$$

The resultant F of the actions of A and B on C is, by symmetry, parallel to the common axis; the work Fx, performed during the displacement x, is equal to the variation of energy. We get from this

$$F = \alpha(V_1 - V_2)\left[V - \frac{1}{2}(V_1 + V_2)\right].$$

We may, indeed, express the coefficient α in functions of the data of the problem. We know, in fact (80), in the case of two unlimited concentric cylinders, the radii of which are R and R_1 and the potentials V and V_1, that the charge of the inner cylinder for the length x is

$$M = \frac{(V - V_1)x}{2}\,\frac{1}{l.\dfrac{R_1}{R}}.$$

From this we get

$$\alpha = \frac{1}{2l.\dfrac{R_1}{R}},$$

and therefore

$$F = \frac{V_1 - V_2}{2l.\dfrac{R_1}{R}}\left[V - \frac{1}{2}(V_1 + V_2)\right].$$

CHAPTER VI.

ON DIELECTRICS.

99. FUNCTION OF THE DIELECTRICAL MEDIUM. — We have hitherto reasoned on the hypothesis that the actions between electrified bodies take place at a distance, and have considered the dielectric as an inert medium, through which the forces act, but as destitute itself of any active properties.

It appears now to be well proved that heat is a vibratory motion, the propagation of which takes place through the intervention of an elastic medium. Now, we have seen that the problem of electrical equilibrium, and that of heat in the permanent state, are characterized by the same mathematical properties.

May we not then suppose that the analogy in the two cases is closer; that it may be followed into the mechanism of the elementary actions; and that there is no other difference in the two orders of phenomena than that which we ourselves introduce into the physical interpretation of the laws? If this is the case, it should be possible to explain the production of the electrical forces by the action of the medium only.

Such is the idea which Faraday sought to elucidate, and which constantly guided him in his researches. This is not the place to attempt to prove, or to disprove, the exactitude of one or the other of these points of view, but simply to show their equivalence in explaining phenomena.

We shall commence by establishing some theorems on the relations between forces and electrostatic pressure.

100. EXPRESSION OF FORCE AS A PRESSURE.—We have already considered as evident, *that the action which is exerted on a conductor is the resultant of the electrical pressures on the whole of its surface;* but it may be useful to consider this theorem from another point of view.

The pressure, on unit surface, at a point of the conductor where the density is σ, and the force F, has the value

$$p = 2\pi\sigma^2 = \frac{1}{8\pi} F^2 = \frac{1}{2} F\sigma,$$

and this pressure is always directed outwards, whatever be the sign of the electricity.

For each element of surface, the pressure pdS is the resultant of the actions exerted, on the mass σdS of this element, by all the masses external to the conductor, and by those on its own surface. The resultant of all the pressures for the entire surface, is the resultant of the actions exerted on this conductor, both by the external masses and by its own electricity. But the resultant of the actions which the various masses of the conductor exert one upon the other is almost null; for as there is equilibrium, those masses may be regarded as fixed on the conductor, and in this case, the elementary forces taken in pairs neutralise each other; the resultant of the pressures is then simply equal to the resultant of the actions of the external masses.

101. *When an electrical system is surrounded by an equipotential surface* S_1, *the action exerted on this system is the resultant of the pressures which would be exerted on a layer equal to the total charge of the system, in equilibrium on the surface* S_1.

Let us suppose that an equipotential surface S_1 divides all the acting masses into two systems, an internal one M_1, and an external one M_2. We have seen that for points external to S_1, the internal masses may be replaced by a layer of the same total mass M_1 in equilibrium on the surface. Conversely, the external system M_2 will act on this layer M_1 fixed on the surface S_1, as it would act on the internal masses, supposed to be connected with each other, so as to form a rigid system.

But, from the foregoing remark, the action of external masses upon the layer S_1, and therefore on the system M_1 of the internal masses, is no more than the resultant of the electrostatic pressures of this layer.

As the total action of the system M_1 on all the external bodies is equal, and of opposite sign to the force which this system experiences, it is also seen that the action of the system M_1, on external bodies, is equal to the resultant of the elementary pressures on the surface S_1, each of them being counted towards the inside.

102. *The reciprocal action of two systems* M_1 *and* M_2 *is equal to the action of two layers* $+M_1$ *and* $-M_1$ *distributed on the two equipotential surfaces* S_1 *and* S_2, *which include* M_1 *and leave* M_2 *outside.*

For, consider a second equipotential surface S_2 (Fig. 22) which includes M_1, and again leaves the system M_2 entirely outside.

Let us arrange a layer $+M_1$ in equilibrium on the surface S_1, and a layer $-M_1$ in equilibrium on S_2; the layer on S_1 may replace the internal system $+M_1$ for all points external to S_1; and the layer on S_2 is equivalent to the external system M_2 for all points on the surface S_2.

The system of these two layers gives, moreover, a constant potential $V_1 - V_2$ inside S_1, and a zero potential outside S_2; and, finally, a potential varying from $V_1 - V_2$ to zero in the intermediate space. The electrical force is therefore everywhere zero, excepting in this space, where it retains the same value at all points, either for the two primitive systems M_1 and M_2, or for the equivalent layers distributed on the surfaces S_1 and S_2.

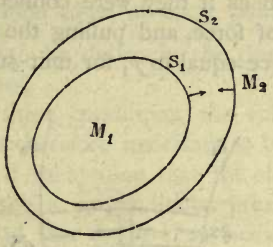

Fig. 22.

The action of the electrified surface S_1 on the layer S_2 is thus the same as on the system M_2; that of S_2, the same on S_1 as upon M_1; the reciprocal actions of the electrified surfaces S_1 and S_2 are thus the same as those of the two primitive systems M_1 and M_2.

But we know from the preceding theorems, that the actions experienced by the surfaces S_1 and S_2, are merely the resultants of the electrical pressures $p_1 dS_1$ and $p_2 dS_2$ which are exerted on the elements of these surfaces. If F_1 and F_2 are the electrical forces in the medium, the values of these pressures near the elements in question are

$$p_1 dS_1 = \frac{1}{8\pi} F_1^2 dS_1,$$

$$p_2 dS_2 = \frac{1}{8\pi} F_2^2 dS_2;$$

the first are directed outside the surface S_1, and the second inside

the surface S_2; and the resultants of these two systems of perpendicular forces $p_1 dS_1$ and $p_2 dS_2$ are equal, and of opposite signs, as they represent action and reaction.

103. The real force which acts between the two electrified surfaces S_1 and S_2 may be regarded as arising from the elementary actions, which are exerted directly and at a distance, between the different electrical masses which cover them, taken in pairs. This is the hypothesis which, up to the present, has formed the basis of all our calculations. But it may also be assumed that this action is transmitted through the surrounding medium in virtue of a special elasticity, as Faraday believed. Regarding it from this point of view, we shall proceed to investigate the mechanical conditions which the intervening medium ought then to satisfy.

For this purpose, let us consider an orthogonal tube between the two surfaces S_1 and S_2. The flow of forces issues from dS_1 (Fig. 23), and is absorbed at dS_2, and the two elements dS_1 and dS_2 are exactly in the same condition as if they were connected by elastic threads parallel to the lines of force, and pulling the two elements towards each other, with a force equal to p_1 for unit surface on dS, and to p_2 on dS_2.

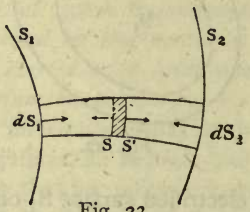

Fig. 23.

Let us take in this tube a volume-element bounded by two infinitely near equipotential surfaces S and S_1 at a distance of dn from each other, and let us suppose this to become solidified. This element must be regarded as subjected to two tensions pulling its bases outwards, and the resultant of which is

$$dR = p'dS' - pdS = \frac{1}{8\pi}(F'^2 dS' - F^2 dS).$$

As from the properties of tubes of force

$$FdS = F'dS',$$

we have

$$dR = \frac{1}{8\pi} FdS\,(F' - F).$$

The two surfaces being infinitely near, we may write

$$F' - F = \frac{dF}{dn} dn;$$

which gives

$$dR = \frac{1}{8\pi} dS dn F \frac{dF}{dn} = \frac{1}{2} dS dn \frac{d}{dn}\left(\frac{F^2}{8\pi}\right) = dS dn \frac{1}{2}\frac{dp}{dn}.$$

Calling R the action relative to unit volume of the dielectric at a point, we have

$$R = \frac{1}{2}\frac{dp}{dn}.$$

The result is accordingly the same as if the forces were exerted on the dielectric itself, and as if the force for unit volume were determined by a potential equal at every point to $-\frac{p}{2}$; hence the volume element tends to be drawn in the direction towards which the function p increases.

104. But, under these conditions, the volume-element cannot be in equilibrium; it is therefore necessary to bring other forces into play. It is sufficient if we assume that the element experiences at each point of its surface a perpendicular pressure p_1 analogous to hydrostatic pressure, and that the level surfaces corresponding to the pressure p_1 coincide with the electrical equipotential surfaces. The volume element will experience an upward pressure.

$$-dS\frac{dp_1}{dn} dn = -dv\frac{dp_1}{dn},$$

which will balance the resultant pressure R if

$$\frac{1}{2}\frac{dp}{dn} - \frac{dp_1}{dn} = 0,$$

or

$$p_1 = \frac{1}{2} p.$$

The true action on the volume-element $dSdn$ consists then of a *pressure* $p_1 = \frac{1}{2}p$ acting on the whole surface, and of a *tension* p on the

bases. This amounts to saying that the lateral surface experiences a pressure $p_1 = \frac{p}{2}$, and the bases a tension equal to the difference $p_1 - p$, that is to say equal to $\frac{p}{2}$.

105. Tension and Repulsion of Lines of Force.—If we consider a layer bounded by two electrified equipotential surfaces, on which we assume there are electrical masses capable of replacing the action of bodies external to the layer, the two surfaces will attract each other with a force equal to the general resultant of the tensions. Dividing this layer into two by an orthogonal surface, a *repulsion* will come into play, between the two portions, equal to the resultant of the lateral pressures.

It is easy to extend these considerations to the case in which the second equipotential surface does not envelope the first.

We may then suppose that *conductors are connected with each other by electric threads stretched along the lines of force, and which repel each other.* This material representation of the phenomena is a useful guide in a great number of applications.

106.—The foregoing properties are the mathematical translation of the idea which Faraday formed for himself of the state of dielectrics, and which he himself summed up in the two following paragraphs of his *Experimental Researches*, Series XI., § 1297–1298 :—

"The direct inductive force which may be conceived to be exerted in lines between two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines, (1224); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction (1304).

" Induction appears to consist in a certain polarized state of the particles, into which they are thrown by the electrified body sustaining the action, the particles assuming positive and negative points or parts which are symmetrically arranged with respect to each other and the inducting surfaces or particles. The state must be a forced one, for it is originated and sustained only by force, and sinks to the normal or quiescent state when that force is removed. It can be *continued* only in insulators by the same portion of electricity, because they only can retain this state of the particles."

107. Energy of the Dielectric Medium.—From this point of view, the whole energy of the electrical system must reside in the dielectrical medium, and it is easy to calculate its value at any point.

The total energy of a system is (91)

$$W = \frac{1}{2}\sum' mV, \qquad \text{or} \qquad W = \frac{1}{2}\int V\rho dv.$$

Replacing the density by its value deduced from Poisson's equation

$$\Delta V + 4\pi\rho = 0,$$

we get

$$W = -\frac{1}{8\pi}\int V\Delta V dv.$$

Hitherto the energy has been determined as a function of the electrical masses themselves. In order to vary the signification, we may apply Green's formula (33)

$$\int V\Delta V dv = \int V\frac{\partial V}{\partial n}dS - \int F^2 dv$$

to the volume bounded by a sphere of very large radius r which includes the electrical system we are considering. The first term of the second member should be extended to the surface of this sphere. The potential V, as we recede, tends to become inversely as r; the factor $\frac{\partial V}{\partial n}$ represents the perpendicular component of the force, and becomes inversely as r^2. As the surface itself is proportional to r^2, this integral is inversely as r, and tends towards zero.

The second member reduces then to the second term, and we have, for the expression of the energy,

$$W = \frac{1}{8\pi}\int F^2 dv.$$

It appears from this that the energy of the system is the same as if each volume element of the medium had a quantity of energy $\frac{1}{8\pi}F^2 dv$. The energy w for unit volume is accordingly

$$w = \frac{1}{8\pi}F^2 = p.$$

Hence *the energy for unit volume is equal at every point to the electrostatic pressure.*

108. Specific Inductive Capacity.—If the dielectric does play this essential part in phenomena, it is not likely that all media behave in exactly the same manner.

We know, in fact, since Franklin's experiments, that the nature of the glass is of great importance in the construction of electrical batteries. Cavendish had already made a great number of experiments to determine directly the comparative effect of various substances used as insulators in condensers, but his experiments were unpublished and unknown at the time when Faraday published his important researches.

Faraday connected the coatings of two spherical Leyden jars of the same dimensions, in one of which the insulating layer of air had been replaced by a solid dielectric such as melted sulphur or resin; he thus found that when a definite charge of electricity was imparted to this system of conductors it did not divide equally between the two jars. That in which the dielectric was solid, took the larger charge.

This is a general phenomenon, and falls under a very simple law. The charge, acquired by a closed condenser, with a solid or liquid dielectric, is in a constant ratio with the charge which it would take, for the same difference of potential, if the dielectric were replaced by a layer of air.

Experiment shows, in fact, that air and gases, even when moist, behave in virtually the same manner, whatever be the pressure and temperature. If the nature of the gas does exert an appreciable influence, to which we shall subsequently refer, it may be neglected in practice.

The ratio thus determined, is what Faraday calls the *specific inductive capacity* of the dielectric. It is, as we see, the number by which the capacity of an air-condenser must be multiplied, to give that of the same condenser, in which the layer of air has been replaced by the dielectric in question.

109. Electrical Absorption. — The determination of this constant offers considerable difficulties for most substances, owing to the occurrence of a phenomenon, to which Faraday gave the name of *electrical absorption*, and which is due to the same cause as the residual charge of condensers. The capacity of a condenser, in which the dielectric is solid, appears as a function of the time; it increases and seems to tend towards a limit, in proportion as the duration of the charge increases. Conversely, when the condenser

is discharged, the disposable electricity which disappears in the discharge is sometimes far below the whole of that which it possesses; it is known moreover that we can successively obtain a greater or less number of discharges of decreasing intensity.

It appears difficult in the present state of science to account for this phenomenon. Everything seems to point to its being due to a progressive change in the structure of the dielectric, to a particular deformation under the influence of causes which produce polarization; a deformation which becomes permanent as in an imperfectly elastic body, and after which the body does not immediately revert to its original state when the cause has ceased to act.

This view of the matter is confirmed by the facts that all the circumstances which, in the case of a mechanical deformation, favour the return of a body to the normal state—such as blows, rapid variations of temperature, and the like, appear also to accelerate the disappearance of the residual charge, and its return to the neutral state.

110. Polarization of the Dielectric.—Although Faraday's experiment is incompetent to settle the question of actions at a distance, it shows unequivocally the part played by the medium in electrical phenomena. We are thereby led to assume that, in electrical induction, the medium acquires a state of polarization analogous to that observed in soft iron when under the influence of a magnet.

In order to explain magnetism, Poisson made a hypothesis which was transferred to the study of electrical phenomena by Mossotti, and then adopted by Faraday. This hypothesis consists in assuming that the magnetic medium, or the dielectric, is made up of particles, which may be spherical for instance, which are absolute conductors, and are disseminated in a non-conducting medium.

"If the space round a charged globe were filled with a mixture of an insulating dielectric, as oil of turpentine or air, and small globular conductors as shot, the latter being at a little distance from each other so as to be insulated, then these would in their condition and action exactly resemble what I consider to be the condition and action of the particles of the insulating dielectric itself. If the globe were charged, these little conductors would all be polar; if the globe were discharged, they would all return to their normal state to be polarized again upon the recharging of the globe." (Faraday, *Experimental Researches*, Series xiv., § 1679.)

Sir W. Thomson has shown that, without making any hypothesis as to the constitution of the medium, it is sufficient to assume that

each volume-element is changed by induction into a small magnet, which may, indeed, be considered as an experimental fact. In this way all the mathematical consequences of Poisson's hypothesis may be deduced.

111. Definition of Dielectric.—A dielectric placed in a field becomes polarized, and the algebraical sum of the masses which form the charge is always null. We know further (59) that, whatever be the condition of an electrified body, the action which it exerts upon an external point is equal to that of a layer of the same total mass as its own, distributed on the surface according to a certain law; in the present case the equivalent layer is formed of two sheets having equal masses and opposite signs.

According to the theory of magnetic induction, which we shall afterwards explain, the action of this layer replaces the effect of polarization, not only for external, but also for internal points. The distribution is determined by the condition, that at two adjacent points, one in air or rather in vacuum, and the other inside the dielectric, the components of the force, perpendicular to the bounding surface, shall be in a constant ratio μ, so that, if F'_n and F_n are the perpendicular components in air and in the dielectric, taken in the same direction, we have

$$\frac{F_n}{F'_n} = \mu, \quad \text{or} \quad F_n = \mu F'_n.$$

Without attempting, for the moment, to examine thoroughly the intimate nature of the phenomenon, we may regard this equation (1) as defining the function of a certain class of bodies, to which experiment shows that the dielectrics, such as we know them, must belong.

We have seen (39) that on both sides of an electrified surface the components of the forces parallel to the surface are equal, and that the difference between the perpendicular components is proportional to the density of the layer,

$$F'_n - F_n = 4\pi\sigma.$$

From which, agreeing to count as positive the perpendicular components on the side of the dielectric, is deduced

$$-\sigma = \frac{1}{4\pi}(F_n - F'_n) = \left(1 - \frac{1}{\mu}\right)\frac{F_n}{4\pi}.$$

Poisson's hypothesis amounts, in short, to supposing that, on the surface of the dielectric, there is a *fictive* layer the density of which σ satisfies this condition.

112. This result may be exhibited under another form. On both sides of an element PP′ or dS of the surface of the dielectric (Fig. 24) let us draw two tubes of force, and let them terminate in two orthogonal bases dS_1 and dS'_1, one in air and the other in the dielectric, and just far enough apart to comprise between them the layer σdS. The flow of force which enters by the base dS_1 is $F dS_1 = F_n dS$; that which emerges by the base dS'_1 is $F' dS_1 = F'_n dS$;

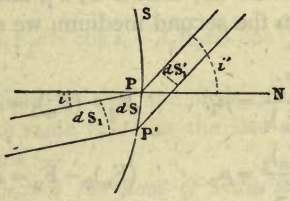

Fig. 24.

the rate of variation of the flow is then equal to $(F_n - F'_n)\, dS$, or from equation (1) which defines the dielectric, to $\left(1 - \frac{1}{\mu}\right)$; it corresponds to a mass of electricity σdS such that

$$\left(1 - \frac{1}{\mu}\right) F_n = -4\pi\sigma.$$

The effect is therefore the same as if a constant fraction of the flow of force were absorbed or emitted by the fictive layer on the surface; the value of this fraction is $1 - \frac{1}{\mu}$.

113. Refraction of the Flow of Forces.—The tangential components being the same in the two media, if i and i' are the angles which the forces F and F′ make with the perpendicular N to the surface S, the expressions

$$F \cos i = \mu F' \cos i',$$
$$F \sin i = F' \sin i',$$

give the equation

$$\tan i = \frac{1}{\mu} \tan i',$$

which expresses what may be called the law of refraction of the force, or of the flow at the moment at which the force passes from air into a liquid or solid dielectric.

114. More generally, let us suppose that the surface S separates two dielectrics, solids or liquids, whose specific inductive capacities are respectively equal to μ_1 and μ_2. If the surface is replaced by an infinitely thin layer of air, then if F, F_1, and F_2 are the forces in air, in the first, and in the second medium, we shall have

$$\frac{F_n}{(F_n)_1} = \mu_1 \qquad F_n - (F_n)_1 = 4\pi\sigma_1,$$

$$\frac{(F_n)_2}{F_n} = \mu_2 \qquad (F_n)_2 - F_n = 4\pi\sigma_2,$$

from which is deduced

$$\frac{(F_n)_1}{(F_n)_2} = \frac{\mu_2}{\mu_1},$$

and

$$\mu_1(F_n)_1 = \mu_2(F_n)_2. \tag{2}$$

The fictive layer is determined by the equation

$$(F_n)_2 - (F_n)_1 = 4\mu\sigma = 4\pi(\sigma_1 + \sigma_2),$$

putting $\sigma = \sigma_1 + \sigma_2$. Lastly, the law of refraction gives for the angles i_1 and i_2 of the forces, with the perpendicular on both sides of the surface, the ratio

$$\frac{\tan i_1}{\mu_1} = \frac{\tan i_2}{\mu_2}.$$

115. TUBES AND FLOW OF INDUCTION.—Let us agree to apply the term *induction at a point, to the product of the force* F *by the specific inductive capacity* μ *of the substance*, and the term *quantity*

or *flow of induction* across a surface-element, to the product of this element by the perpendicular component of induction; the preceding results may then be expressed in a very simple manner.

We observe, in the first place, that in gaseous media, or at any rate in a vacuum, $\mu = 1$; the induction and the force have the same numerical expression, and tubes of force are identical with tubes of induction just as are the two kinds of flow.

In the case of continuous media whose specific inductive capacities are μ_1 and μ_2, the ratio of the perpendicular components

$$\mu_1 F_1 \cos i_1 = \mu_2 F_2 \cos i_2$$

gives

$$\mu_1 F_1 dS \cos i_1 = \mu_2 F_2 dS \cos i_2,$$

an equation which signifies that the flow of induction across the element dS retains the same value in the two media. We are thus led to the following law:—

In a tube of induction the flow of induction retains a constant value, whatever be the dielectric media which it traverses, so long as it does not meet a really electrified body. This law merges into that of the *conservation of the flow of force* when we are only considering a single medium.

If the tube encounters a mass of electricity m situate in the dielectric medium, we may always look upon this mass as separated from the dielectric by a layer of air; in this layer the flow of induction merges into the flow of force; as the latter varies by $4\pi m$, this is also the case with the flow of induction, in virtue of the preceding theorem.

116. Characteristic Equations of Induction.—If we apply this theorem to a volume-element $dxdydz$ in a dielectric whose specific capacity is μ, at a point where the real density of electrification is ρ, we obtain the following equation, analogous to that of Poisson.

$$(3) \qquad \frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial V}{\partial z}\right) + 4\pi\rho = 0.$$

As, for the present, we are only considering isotropic media, the factor μ is constant, and this expression reduces to

$$\mu\Delta V + 4\pi\rho = 0.$$

At the bounding surface of two media we shall always have to distinguish two densities; the density σ of the fictive layer, which must be assumed at the bounding surface of the dielectrics in order that opposite any point, outside this surface, there may be the effect equivalent to their internal polarization; there is also the density σ' of the true layer, which might have been developed, by friction for instance, on this same surface.

We shall thus have for the bounding surface of the two media the equations

$$(F_n)_2 - (F_n)_1 = 4\pi(\sigma + \sigma'),$$

$$\mu_2(F_n)_2 - \mu_1(F_n)_1 = 4\pi\sigma'.$$

Agreeing to count in each medium the perpendiculars from the surface, and calling V_1 and V_2 the values of the potential on the two dielectrics respectively, these equations may be written

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$$\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2} + 4\pi(\sigma + \sigma') = 0, \tag{4}$$

$$\mu_1 \frac{\partial V_1}{\partial n_1} + \mu_2 \frac{\partial V_2}{\partial n_2} + 4\pi\sigma' = 0. \tag{5}$$

The inductive capacity μ is always positive and greater than unity; in conductors it may be regarded as equal to infinity.

117. In the case in which the dielectrics have not received electricity either in the interior or on the surface, these equations reduce to

$$\Delta V = 0,$$

$$\frac{\partial V_1}{\partial n_1} + \frac{\partial V_2}{\partial n_2} + 4\pi\sigma = 0,$$

$$\mu_1 \frac{\partial V_1}{\partial n_1} + \mu_2 \frac{\partial V_2}{\partial n_2} = 0.$$

from which we have

$$\sigma = \frac{\mu_1 - \mu_2}{4\pi\mu_2} \frac{\partial V_1}{\partial n_1}.$$

118. Observations on the Fictive Layer.—Although the layer of density σ is a fictive layer, it must be noticed that if, while the dielectric is under induction, its surface is brought by any means to the neutral state, by moving along it a flame connected with the earth for instance, and if the sources of induction are removed, a real layer of density σ will be found on this surface.

This observation enables us to explain the phenomena exhibited by certain bodies; for instance, uniaxial pyroelectrical crystals such as tourmaline. We need only suppose that the normal state of these bodies is analogous to that which dielectrics acquire under the influence of electrical forces—in other words, that they are *naturally polarized*, and that *their state of polarization is a function of the temperature.*

A tourmaline which in appearance is neutral, is a tourmaline which, in virtue of its polarization, would exert on the outside the same forces as a layer of total mass zero, and density σ, distributed on the surface, but which from any causes—for instance, losses by contact with the surrounding medium—has become covered by a real layer of density $-\sigma$, which for any external point neutralises the effect of internal polarization. If the temperature of the tourmaline alters, its internal condition may be changed without modifying the layer developed on the surface; equilibrium is broken, and could only be restored more or less slowly under the action of causes which had brought about the previous neutralisation; the effect observed under these conditions is the difference between the actions of the fictive and of the real layer.

119. Charges of Two Corresponding Elements. — The theorem of corresponding elements (36) also holds when the two conductors are placed in different media. This will be clear if we remember that we can always imagine the conductor separated from the dielectric by an infinitely thin layer of air, between the surface of the conductor itself and an infinitely near equipotential surface.

Let A and B be the two conductors, μ_1 and μ_2 the inductive capacities of the dielectrics with which they are respectively in contact. If the bounding surface S of the two dielectrics has no real electrical layer, the flow of induction is the same throughout the whole extent of an orthogonal tube which cuts, on the conductors and on this surface, the elements $d\mathrm{S}_a$, $d\mathrm{S}_b$, and $d\mathrm{S}$.

The force which in air, near the first conductor, would be F_a, becomes $\mathrm{F}_1 = \frac{\mathrm{F}_a}{\mu_1}$ in the dielectric.

The apparent density σ'_a on the conductor—that is to say, that

which would give the force F_1 by the ordinary ratio $4\mu\sigma'_a = F_1$, is equal to the algebraical sum of the real density σ_a of the conductor and of the fictive density σ_1 at the surface of the dielectrics; from this we get

$$\sigma'_a = \sigma_a + \sigma_1 = \frac{\sigma_a}{\mu_1},$$
$$\sigma_1 = -\sigma_a\left(1 - \frac{1}{\mu_1}\right).$$

In like manner, on the conductor B we have

$$\sigma'_b = \sigma_b + \sigma_2 = \frac{\sigma_b}{\mu_2},$$
$$\sigma_2 = -\sigma_b\left(1 - \frac{1}{\mu_2}\right).$$

If the surface S has a real layer of density σ', the fictive layer having the density σ, the perpendicular forces on both sides satisfy the equation

$$(F_n)_2 - (F_n)_1 = 4\pi(\sigma + \sigma').$$

We have further, in the two media respectively,

$$F_1 dS_a = (F_n)_1 dS,$$
$$F_2 dS = (F_n)_2 dS.$$

Replacing the forces F_1 and F_2 by their values $4\pi\sigma'_a$ and $-4\pi\sigma'_b$,

$$\sigma'_a dS_a - \sigma'_b dS_b + (\sigma + \sigma')dS = 0.$$

This equation expresses that *the algebraical sum of the apparent charges of corresponding elements of the two conductors, is equal and of opposite sign to the total charge of the corresponding element of the bounding surface of the two dielectrics.*

120. ENERGY OF A SYSTEM IN THE CASE OF ANY GIVEN DIELECTRICS.—The general expression of energy is, as we have seen (107),

$$W = \frac{1}{2}\sum mV = \frac{1}{2}\int V\rho dv.$$

The equation

$$\mu\Delta V + 4\pi\rho = 0$$

gives

$$W = -\frac{1}{8\pi}\int \mu V \Delta V dv.$$

From Green's formula, and the remark already made (107), this expression reduces to

$$W = \frac{1}{8\pi}\int \mu F^2 dv.$$

We have then, for the energy of unit volume,

$$w = \frac{1}{8\pi}\mu F^2.$$

121. COMPARISON WITH THERMAL PHENOMENA.—Let us resume the comparison of the problem of electrical equilibrium with that of the propagation of heat. We have seen that between two identical level surfaces, if the coefficient of conductivity is equal to unity, the flow of heat in the first is numerically equal to the flow of force in the second; if the coefficient of conductivity is k, the flow of heat is k times the flow of electrical force.

Let us now consider two correlative systems, one electrical and the other thermal, each formed of two media separated by the same surface S, and such that the equipotential surfaces of the one, coincide with the isothermal surfaces of the other If k_1 and k_2 are the coefficients of conductivity of the two media, the flow of heat across an element dS at the bounding surface, in the first medium, is

$$k_1 \frac{\partial V_1}{\partial n_1} dS,$$

and in the second

$$-k_2 \frac{\partial V_2}{\partial n_2} dS.$$

As thermal equilibrium is supposed to have been attained, these two flows are equal, and we have

$$k_1 \frac{\partial V_1}{\partial n_1} + k_2 \frac{\partial V_2}{\partial n_2} = 0.$$

The electrical system gives, in the same way,

$$\mu_1 \frac{\partial V_1}{\partial n_1} + \mu_2 \frac{\partial V_2}{\partial n_2} = 0.$$

From this we deduce

$$\frac{\mu_1}{\mu_2} = \frac{k_1}{k_2}.$$

The flows of induction are therefore proportional to the flows of heat; the specific inductive capacity playing the same part in the electrical problem as the coefficient of conductivity in the thermal problem.

122. Change of Potential Produced by Interposing a Dielectric.—If we introduce a conductor into an electrical field due to insulated and electrified conductors, the presence of this new body has the effect of diminishing the initial energy of the system. The introduction of a solid or liquid dielectric produces the same effect to a lesser degree.

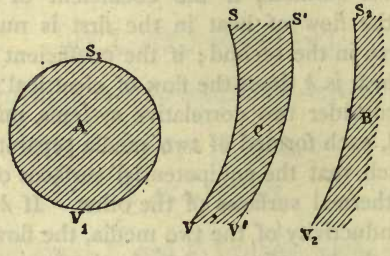

Fig. 25.

As an instance of this, let us consider the case of conductor A (Fig. 25), charged with a quantity M_1 of electricity, and situate inside a closed conductor B kept at a constant potential V_2.

Equilibrium being established, let us fix the electrical masses on A and B and introduce into the interval a dielectric layer C, of inductive capacity μ, the internal and external surfaces of which, S and S', are equipotential surfaces belonging to the primitive system, where the potentials were respectively V and V'. It is easy to see that equilibrium is not disturbed when we distribute on the surfaces S and S' electrical charges, $-M$ and $+M$, identical with

those which would be produced if this medium were a conductor, and that the primitive charge of A had been replaced by M.

The charges $+M$ on A, $-M$ and $+M$ on C, and $-M$ on B do, in fact, establish constant potentials on the bodies A, B, and C; on the other hand, the equal and opposite layers, $+M_1$ and $-M_1$, produce constant potentials on the conductors A and B, so that they are in equilibrium. The form of the equipotential surfaces intermediate to the conductors A and B is not modified, and the direction of the force remains everywhere the same.

From the surface S_2 to the surface S' the increase of potential is the same as if the layer C did not exist; the variation also remains the same from S to S_1.

In order to establish the condition relative to the dielectric, let us consider an orthogonal tube which cuts on the surfaces S_1 and S, the elements dS_1 and dS, the densities on which have the absolute values σ_1 and σ. The flow of force $4\pi\sigma_1 dS_1$ which issues from the element dS_1 is partly absorbed on the element dS, and the fraction lost is

$$\frac{4\pi\sigma dS}{4\pi\sigma_1 dS_1}.$$

As this fraction should be equal to $1-\frac{1}{\mu}$ (112), it follows that the ratio of the charges σdS and $\sigma_1 dS_1$ of the two corresponding elements is also $1-\frac{1}{\mu}$. The condition of equilibrium of the dielectric is then satisfied if we have

$$\frac{M}{M_1}=1-\frac{1}{\mu}.$$

The force having become μ times less between the surfaces S and S', the fall of potential has diminished in the same ratio, so that the total increase of potential in going from B to A is now

$$V'-V_2+\frac{1}{\mu}(V-V')+V_1-V=V_1-V_2-\left(1-\frac{1}{\mu}\right)(V-V').$$

Calling U_1, the new potential of the conductor A, we have

$$U_1-V_2=V_1-V_2-\left(1-\frac{1}{\mu}\right)(V-V').$$

If the body interposed were a conductor, the loss of potential of the conductor A would be $V - V'$. The effect of introducing the dielectric has been to lower the potential on the conductor A, and the fall is a fraction equal to $1 - \frac{1}{\mu}$ of what would be produced by a conductor of the same dimensions as the dielectric. This simple result is, however, peculiar to the conditions chosen; it would not be the same if the dielectric were not bounded by the level surfaces of the original system.

123. When the dielectric occupies the whole space between the conductors A and B, so as to form a closed condenser, we have $V = V_1$, $V' = V_2$, and

$$U_1 - V_2 = \frac{1}{\mu}(V_1 - V_2).$$

For the same charge the difference of potentials has become μ times less, by substituting for the layer of air a dielectric whose specific inductive capacity is equal to μ. In other words, the capacity of the system has become μ times as great. This is just Faraday's experiment.

The above remark (119) gives directly the latter results. By interposing a dielectric of the specific inductive capacity μ, in the space which separates A and B, the form of the equipotential surface is not modified, but the apparent density at each point becomes μ times less than the real density; the effect is the same as if the system, retaining its original capacity, had received a charge μ times smaller.

124. We may represent to ourselves the preceding phenomenon in still another manner.

Let us suppose that the dielectric comprised between the conductors A and B (Fig. 26) is divided into an odd number of infinitely thin laminæ α, β, α', β'.... by equipotential surfaces of the original system, so that the variation in potential is the same in all the layers α, α'... respectively, as in the layers β, β'... and that we have, therefore,

$$(6) \qquad \begin{aligned} V_1 - V' &= V'' - V''' = \ldots\ldots = H_\alpha, \\ V' - V'' &= V''' - V^{IV} = \ldots\ldots = H_\beta. \end{aligned}$$

Let us finally place on each of these surfaces masses equal in absolute value to those on the surfaces S_1 and S_2 of the conductors

alternately positive and negative, + M on the odd surfaces, and − M on the even surfaces, each of these layers being in equilibrium under the action of the original conductors A and B.

It is clear that the system thus obtained is in equilibrium. The force is not modified in all the odd laminæ α, α', α''.., but it is null in all the even ones β, β', β''... and the potential has a constant value in each of these laminæ. It is as if all the even laminæ were replaced by conducting layers.

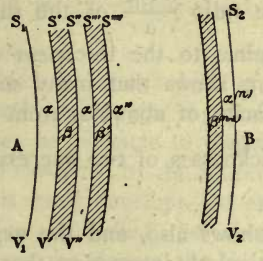

Fig. 26.

This operation has lowered the difference of potential between A and B. The difference of potential of the surface S_1 to S'', which was originally $V_1 - V''$, has, in fact, become $V_1 - V'$, and we have

$$V_1 - V' = \frac{\alpha}{\alpha + \beta}(V_1 - V'').$$

In like manner, the original difference of potenial $V - V^{IV}$, between S'' and S^{IV} is reduced to

$$V' - V''' = \frac{\alpha'}{\alpha' + \beta'}(V'' - V^{IV}).$$

The ratios

$$\frac{\alpha}{\alpha + \beta}, \quad \frac{\alpha'}{\alpha' + \beta'} \ldots\ldots,$$

are the same throughout the whole thickness of the dielectric, and we may write

$$\frac{\alpha}{\alpha + \beta} = \frac{\alpha'}{\alpha' + \beta'} \ldots\ldots = \frac{1}{\mu}.$$

Calculating, in this way, from layer to layer, we see that from the surface S_1 to the surface S_2, the potential varies μ times less than in

the original state, so that by interposing the dielectric the difference of potential of two surfaces has become

$$U_1 - U_2 = \frac{1}{\mu}(V_1 - V_2),$$

which gives the same result as the preceding.

Looked at in this way, the coefficient μ acquires a physical signification; it is the ratio $\frac{\alpha+\beta}{\alpha}$ of the sum of the thicknesses of two successive laminæ to the thickness of that one which is odd. Now, experiment shows that many solid dielectrics have a specific inductive capacity of about 2, from which it follows that the ratio $\frac{\alpha}{\beta}$ of the thicknesses of two successive laminæ would be sensibly equal to unity.

125. Experiment shows also, and the experiments of Gaugain on this subject are particularly interesting, that the specific inductive capacity varies with the time. It has first a minimum value at the moment of charge; it then increases rapidly and afterwards more slowly, tending then towards a limit. In other words, the potential of the inner coating of a condenser first diminishes rapidly after the charge, and then more slowly.

The force being zero in each of the β, β'..., we see in fact that the positive layers are all impelled outwards and the others inwards, and that in consequence of this mutual action the layers which bound the laminæ α tend to come nearer, which more and more increases the inductive capacity.

Generalising this reasoning, we are led to attribute to conductors an infinitely great specific inductive capacity.

126. MAXWELL'S THEORY OF DISPLACEMENT. — In order to explain the properties of dielectrics and to account for the phenomena by the intervention of the medium only, Maxwell supposed that when a dielectric is submitted to induction a phenomenon is produced equivalent to a *displacement* or *gliding* of electricity in the direction of the induction. For instance, in a Leyden jar whose inner coating is charged positively, and outer negatively, the displacement takes place in the substance of the glass from within outwards.

Any increase of the charge increases the displacement, and corresponds to a current of positive electricity from the inside towards the outside; any diminution, to a current going from the

outside towards the inside; the duration of the current is equal to that of the variation.

The displacement through any surface is the quantity of electricity which traverses it. Let σdS be this quantity; for an element dS of the surface of a conductor the displacement is equal to $\frac{F dS}{4\pi}$, it is therefore equal to the corresponding flow of force divided by 4π. In contact with a dielectric the quantity of electricity has the value $\frac{\mu F dS}{4\pi}$; the displacement is accordingly equal to the quotient of the flow of induction by 4π.

Generally, *the displacement, at any point of a dielectric, is equal to the quotient of the induction by* 4π, *and is parallel to this force.*

A conductor opposes no obstacle to displacement. In a dielectric the displacement is restricted by the action of antagonistic forces which the displacement itself develops—in other words, by a kind of elasticity, which may be called the *electrical elasticity* of the medium. If, by analogy, we denote the ratio of the force to the displacement which it produces, by the term *coefficient of electrical elasticity*, and suppose the medium to be perfectly elastic, it will be seen that the coefficient of electricity is equal to $\frac{4\pi}{\mu}$, and that therefore the specific inductive capacity is inversely proportional to the coefficient of elasticity of the medium.

The displacement produced by induction across the entire mass of the dielectric determines the polarization of the medium and the apparent electrification of the conductors.

Consider a tube of induction between two conductors. Throughout the whole extent of the tube the displacement is constant; every orthogonal section is traversed by the same quantity of electricity. At one end, the displacement is from the conductor towards the dielectric, the corresponding element dS of the conductor is then said to be charged with positive electricity of density σ; at the other end the displacement is from the dielectric towards the conductor, and the corresponding element dS is charged with a density $-\sigma'$. Throughout the whole extent of the tube, if the dielectric is the same, there is no apparent electricity; but this medium is polarized; for, conceive for a moment a portion of a tube comprised between two orthogonal sections: the displacement has taken place in the contrary direction for the two sections, and they would appear oppositely electrified if their electrification were not neutralised by the equal and opposite electrification of the portions of the tube in

contact. If the tube traverses the surface of separation of the two electricities, the displacement is the same in the two media, but the polarization is not the same, and the surface would have an apparent electricity equal to the difference of the electrical layers on the surfaces of the two media in contact.

It is evident that, since the electrification of the conductor is only apparent, all the energy due to the electrification must reside in the medium. It is equal to the work expended in effecting the displacement in a direction opposite that of the elastic forces. From what we have seen (120), the value of this work for unit volume is $\frac{\mu F^2}{8\pi}$ or $\frac{1}{2} \cdot \frac{\mu F}{4\pi} \cdot F$; it is therefore equal to half the product of the electrical force with the displacement.

Maxwell's theory of displacement accounts thus for the properties of the medium in a satisfactory manner. It furnishes a physical interpretation of Faraday's specific inductive capacity; when multiplied by a factor $\frac{1}{4\pi}$, it is the inverse of the coefficient of electrical elasticity of the medium.

It explains Faraday's view that it is not possible to impart an absolute charge of electricity to matter: on this theory, in short, electricity behaves like an incompressible fluid; the quantity which can be contained in a closed surface is invariable, and the production of two quantities of electricity of equal and opposite signs appears to be a consequence of one and the same phenomenon.

In conclusion, it is natural to suppose that if the explanation of electrical phenomena postulates the existence of an incompressible medium, diffused in space, this medium can be none other than the ether to which luminous and thermal phenomena are attributed; this theory enables us to discern a dependence between the two orders of phenomena, the confirmation of which would be one of the most important conquests of physical science.

CHAPTER VII.

PARTICULAR CASES OF EQUILIBRIUM.

127. REPRESENTATION OF THE ELECTRICAL FIELD.—The condition of an electrical field is defined at every point by the direction and magnitude of the force. It may be represented either by *equipotential surfaces* or by *lines of force.*

In the former case, equipotential surfaces are drawn which correspond to the numerical values of the potential 1, 2, 3 .... n, and which, therefore, are such that the transference of unit electricity from any given surface to the next following one, corresponds to a unit of work.

The force, at each point, is perpendicular to the equipotential surface; its mean value F_1 between two consecutive surfaces of the orders n and $n+1$, at a distance of a from each other, is defined by the equation

$$F_1 a = V_n - V_{n+1} = 1.$$

The value of the mean force is therefore inversely as a.

These surfaces may be represented by a graphic method.

Take first the case of a single centre of force, a point charged with a mass m. The potential at the distance r is

$$V = \frac{m}{r};$$

the equation

$$r = \frac{m}{V}$$

determines the radius of the sphere, the potential of which is V. Let V have the values 1, 2, 3... n, and draw the corresponding spheres, we shall have equipotential surfaces, whose potentials correspond to the natural series of numbers.

128. Let us now assume that several centres, of masses m, m', m'' ..... act simultaneously; the resultant potential at a point being the

sum of the potentials relative to each of the centres, it is clear that the points, whose potential is V_p, will be obtained by the intersection of spheres of potentials

$$V_n, V_{n'}, V_{n''} \ldots\ldots,$$

such that

$$n + n' + n'' \ldots\ldots = p.$$

and that the geometrical locus of all these points will be the level surface of potential V_p.

This is a method of general application, and enables us, in theory at least, to determine the equipotential surface of any system whatever.

Their representation in a plane could be completely made only in the case of a system of revolution traced on a meridian plane. The force will always be in the plane of the figure, perpendicular at each point to the meridional section of the equipotential surfaces, and inversely as their distance.

If the system is symmetrical in reference to a plane, we could still have a complete representation of the state of the field in the plane of symmetry In any other case the intersection of a system of equipotential surfaces by any plane will give a series of curves which are equipotential curves; the component of the force along the intersecting plane is perpendicular to the curves at every point, and is inversely as their distance; but the value of the true force is not represented.

129. The lines of force may give an equivalent representation for the field. Such a line, being perpendicular at every point to the equipotential surface, indicates the direction of the force; in order to represent the strength at the same time, we agree to divide the field into tubes of force, such that the flow corresponding to each of them has a constant value, unity for instance.

An equipotential surface being given, it is sufficient to divide it into elements dS such that $F dS = 1$, and to take each of these elements as the base of an orthogonal tube.

The division is an arbitrary one, and in each case that would be chosen which leads to the simplest construction.

130. Uniform Field.—In the case of a *uniform field* all the equipotential surfaces are equidistant planes, perpendicular to the direction of the force. The simplest division consists in drawing two series of planes at right angles to each other, and parallel to the direction of the force. The equipotential surfaces will then be cut out in equal rectangles.

Any section by a plane P, parallel to the direction of the field, will give two systems of equidistant lines of force, which will be the

intersection of the plane of the figure with the two series of planes perpendicular to the equipotential surfaces.

131. FIELD SYMMETRICAL IN REFERENCE TO A PLANE.—For all points of the plane of symmetry, the force is perpendicular to the plane; lines of force may be traced such that the product of the

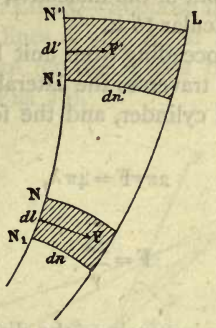

Fig. 27.

force by the distance dl (Fig. 27) of the consecutive lines is constant

$$F dl = F' dl';$$

since we have

$$F dn = F' dn',$$

it follows that

$$\frac{dl}{dn} = \frac{dl'}{dn'}.$$

The curvilinear rectangles $dndl$, $dn'dl'$, formed by the two infinitely near equipotential surfaces L and L', and the two lines of force, are *similar*.

The flow of force will only be determined by taking into account the dimensions of the tube perpendicular to the plane of symmetry. If we assume that these dimensions are everywhere the same, the flows will not be equal except in that case in which all the planes parallel to the plane of the figure are identical. This is the case of a cylindrical distribution, all the bodies of the system being parallel cylinders; it corresponds to the problem of the *propagation in a plane* in the theory of heat. The potential, or the temperature, no longer depend on only two co-ordinates x and y, and Poisson's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 4\pi\sigma = 0,$$

σ being the electrical density on the plane.

We shall proceed to examine this particular case in some detail, less for its own importance than as a useful transition to more complicated problems.

132. CYLINDRICAL SYSTEMS.—Consider a uniformly electrified unlimited line of density λ, that is to say where the charge is λ for unit length. At each point of the dielectric, the force passes through the axis, and is perpendicular to it.

The flow of force proceeding from unit length is equal to $4\pi\lambda$; at a distance r, this flow traverses the lateral surface $2\pi r$ of the corresponding equipotential cylinder, and the force F is defined by the condition

$$2\pi r\mathrm{F} = 4\pi\lambda,$$

where

$$\mathrm{F} = \frac{2\lambda}{r}. \tag{1}$$

The force is therefore inversely as the distance, as we have already seen (80) for cylindrical condensers. The equation

$$\mathrm{F} = -\frac{d\mathrm{V}}{dr} = \frac{2\lambda}{r},$$

gives for the equipotential surfaces

$$\mathrm{V} = -2\lambda l.r + \text{const}, \tag{2}$$

that is to say, a series of concentric cylindrical surfaces. Draw two planes perpendicular to the axis and at the distance ϵ; the mass which they include is $m = \lambda\epsilon$. The flow of force which will pass between these two planes will therefore be $4\pi m$, and it is evident that if we draw through the axis $4\pi m$, planes making equal angles with each other, each of the $4\pi m$ dihedra thus determined constitutes an orthogonal tube in which the flow of force is equal to unity.

Take a plane perpendicular to the axis as the plane of the figure. Let A (Fig. 28) be the trace of the electrified line, and Ax any axis from which we shall count 1, 2, 3..., the traces of the $4\pi m$ planes drawn through the axis. Lastly, let θ be the angle which the straight line, number N, makes with the axis Ax, it is manifest that the flow of force corresponding to the angle θ is

$$\mathrm{Q} = 4\pi m\frac{\theta}{2\pi} = 2m\theta.$$

This flow on the other hand is equal to N units; we have therefore

$$(3) \qquad Q = 2m\theta = N, \quad \text{or} \quad \theta = \frac{N}{2m}.$$

133. Two Parallel Lines.—Suppose that the electrical system consists of two parallel lines A and A′, of densities λ and λ', such that $m = \epsilon\lambda$, $m' = \epsilon\lambda'$; take for the x axis the straight line joining the two lines A and A′ (Fig. 28).

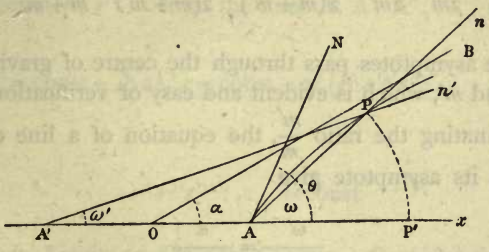

Fig. 28.

Through these two points draw two straight lines An and A′n' of the orders n and n' respectively in reference to the centres A and A′, and making angles ω and ω' with the axis; join their point of intersection P to the axis by any given curve PP′. It is evident that across the cylindrical surface PP′, there is a flow n, or $2m\omega$ from A, and a flow n' from A′, and therefore a total flow equal to $n + n' = N$. The same will be the case with all the points of the curve AP, defined by the points of intersection, two by two, of the straight lines proceeding from A and A′, and such that the sum of their numbers is equal to N; the locus of all these points is evidently a line of force of the order N for the resulting system.

The force near one of the acting masses depends only on this mass, the influence of which predominates. The line of flow of the order N is therefore tangential at A to the right line of order N drawn from this point.

The equation of the curve AP is

$$(4) \qquad n + n' = N,$$

from which, replacing these quantities by their values as functions of the angles,

$$(5) \qquad m\omega + m'\omega' = m\theta.$$

This is the equation of the lines of force drawn from the point

A; an analogous equation will give those which proceed from the point A'. If m and m' are of the same sign, all these lines are unlimited; any one of them—that of the order N, for instance—is an asymptote to a right line making an angle α with the axis Ax; this angle is defined by the condition that the right lines of order n', connected by the ratio (4), are parallel to each other—that is, that we have

$$(6) \qquad \alpha = \frac{n}{2m} = \frac{n'}{2m'} = \frac{n+n'}{2(m+m')} = \frac{N}{2(m+m')} = \frac{m}{m+m'}\theta.$$

All these asymptotes pass through the centre of gravity O of the masses m and m', which is evident and easy of verification.

By eliminating the ratio $\frac{m'}{m}$, the equation of a line of force (5) and that of its asymptote give

$$\frac{\omega'}{\theta-\omega} = \frac{\alpha}{\theta-\alpha}$$

This is the equation of the line of force as a function of the angles which the asymptote, and the tangent at the origin make with the axis Ax.

If r and r' are the distances of a point P to the two lines A and A', the equation of the equipotential surfaces is

$$V = \text{const} - 2\left[\lambda l.r + \lambda' l.r'\right] = \text{const} - 2l.(r^{\lambda} r'^{\lambda'}),$$

from which

$$r^{\lambda} r'^{\lambda'} = \text{const.}$$

134. Several Parallel Lines.—It is evident that this method of construction may be applied to any number of electrified lines A, A', A''... defined as above by the masses m, m', m'',..., on the condition that these lines are parallel and situate in the same plane.

The general equation of the lines of force starting from the centre A of the mass m will be in this case

$$m\omega + m'\omega' + m''\omega'' \ldots\ldots = m\theta,$$

the masses m, m',... may be positive or negative; that of the corresponding asymptote is

$$(m+m'+m''\ldots\ldots)\alpha = m\theta$$

When all the masses are of the same sign, all the lines of force are unlimited. In the contrary case, part of the flow of force issuing from positive masses is absorbed by negative masses.

From the method of numbering adopted, the number of unlimited lines of force is equal to the difference between the number of positive lines and of negative lines.

If the electrified lines A, A', A''...., while still parallel, are no longer in the same plane, the construction of lines of force becomes more complicated. In this case, the value of the potential at a point P at distances r, r', r''... from the lines A, A', A''...., is

$$V = \text{const} - \sum 2\lambda l.r = \text{const} - 2l.(r^{\lambda} r'^{\lambda'} r''^{\lambda''} \ldots\ldots),$$

whence

$$r^{\lambda} r'^{\lambda'} r''^{\lambda''} \ldots\ldots = \text{const.}$$

135. Two Lines of Opposite Signs.—Consider the particular case of two lines electrified oppositely, defined by the masses $+m$

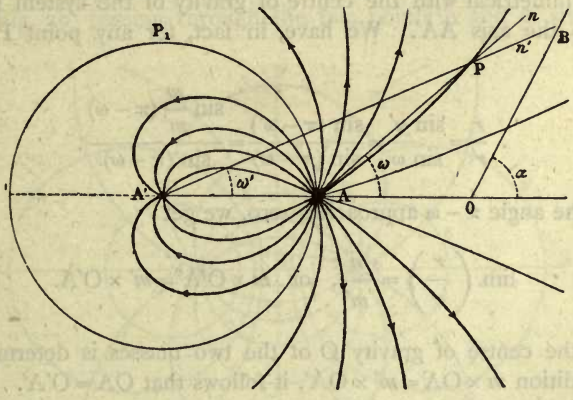

Fig. 29.

and $-m'$, situate at two points A and A' (Fig. 29) at the distance $2a$, and let m be the greater of these masses. The equation of a line of force becomes

$$n - n' = \text{N},$$

from which

$$m\omega - m'\omega' = m\theta;$$

that of the corresponding asymptote is

$$\alpha = \frac{m}{m - m'}\theta.$$

As the angle α cannot become greater than π, there can only be unlimited lines of force for values of θ smaller than

$$\theta_0 = \pi \frac{m - m'}{m}.$$

The line of force AP_1 corresponding to this value θ_0 separates the $m - m'$ lines of force proceeding from A, and which are unlimited, from the m' which are finite and are absorbed at A'. The equation of this limiting line of force is

$$m\omega - m'\omega' = m\theta_0 = (m - m')\pi, \quad \text{or} \quad \pi - \omega' = \frac{m}{m'}(\pi - \omega).$$

This equation is satisfied, for $\omega = \pi$ and $\omega' = \pi$; hence the line meets the axis on the left of the point A', and the point of meeting O' is symmetrical with the centre of gravity of the system in reference to the axis AA'. We have, in fact, for any point P_1 of the curve

$$\frac{r}{r'} = \frac{\sin \omega'}{\sin \omega} = \frac{\sin (\pi - \omega')}{\sin (\pi - \omega)} = \frac{\sin \frac{m}{m'}(\pi - \omega)}{\sin (\pi - \omega)}.$$

If the angle $\pi - \omega$ approaches zero, we get

$$\lim. \left(\frac{r}{r'}\right) = \frac{m}{m'}, \quad \text{or} \quad m \times O'A' = m' \times O'A.$$

As the centre of gravity O of the two masses is determined by the condition $m \times OA = m' \times OA'$, it follows that $OA = O'A'$.

In the case of $m = 2m'$ (Fig. 29) we have

$$\theta_0 = \frac{\pi}{2}, \text{ and } 2\omega - \omega' = \pi,$$

from which we get

$$r' = \frac{r}{2 \sin \frac{\omega'}{2}} = 2a;$$

The limiting line of force is therefore a circumference whose centre is A′, and which passes through the point A.

The equation of the equipotential surfaces is

$$V = \text{const} - 2l.\left(\frac{r^{\lambda}}{r'^{\lambda'}}\right) = \text{const} + 2l.\left(\frac{r'^{\lambda'}}{r^{\lambda}}\right),$$

from which

$$\frac{r'^{\lambda'}}{r^{\lambda}} = \text{const}\, e^{\frac{V}{2l}}.$$

136. Two Equal Lines of Opposite Signs.—If we suppose the two masses equal in absolute values, the equation of the lines of force reduces to

$$\omega - \omega' = \theta,$$

and that of the equipotential surfaces to

$$V = 2\lambda l.\frac{r'}{r}.$$

The former represents segments of the circumference such as ATA′ (Fig. 30) passing through the two points A and A′ and which

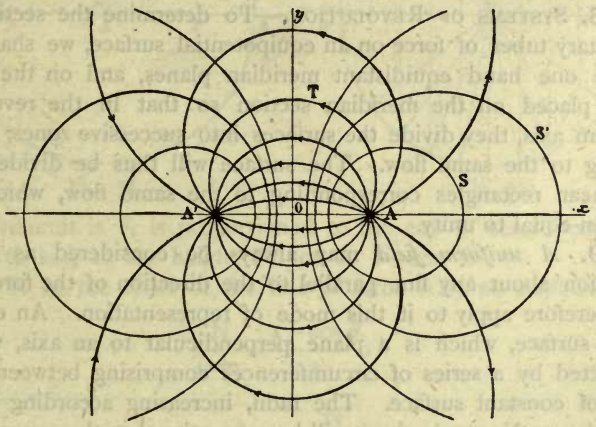

Fig. 30.

may have the angle θ; the second represents circumferences S, S′... having their centres on the right line AA′, and such that the two points are conjugate in reference to each of them.

Considering the two equipotential surfaces S and S′, a layer $+m$

on each unit of length of the cylinder S, and a layer $-m$ on each unit of length of the cylinder S', will replace the action of the two unlimited lines A and A' (61) for all points between the two surfaces; the figure will correspond, in this case, to the problem of a condenser formed of two unlimited excentric cylinders.

137. Let us suppose that the distance $2a$ approximates to zero, but that the density λ varies so that the product $2a\lambda$ remains constant. The potential at the distance r, in a direction which makes the angle ω with the straight line, will be

$$V = 2\lambda l.\left(1 + \frac{2a\cos\omega}{r}\right) = 2\,\frac{2a\lambda\cos\omega}{r}.$$

This equation represents circumferences whose radii vary as the reciprocals of the series of even numbers.

We have in like manner for the lines of force,

$$\omega - \omega' = \frac{2a\sin\omega}{r} = \theta.$$

Thus the radii of the circumferences which represent the lines of force vary also as the reciprocals of the series of even numbers.

138. SYSTEMS OF REVOLUTION.—To determine the sections of elementary tubes of force on an equipotential surface, we shall take on the one hand equidistant meridian planes, and on the other, points placed on the meridian section so that in the revolution about an axis, they divide the surfaces into successive zones, corresponding to the same flow. The surface will thus be divided into curvilinear rectangles corresponding to the same flow, which will be taken equal to unity.

139. *A uniform field* may always be considered as one of revolution about any line parallel to the direction of the force; we may therefore apply to it this mode of representation. An equipotential surface, which is a plane perpendicular to an axis, will be intersected by a series of circumferences comprising between them zones of constant surface. The radii, increasing according to the same law as Newton's rings, will be proportional to the square roots of consecutive numbers. The lines of force will then be represented in the meridian plane by right lines, parallel to the axis, and whose distances from the axis are as the square roots of consecutive whole numbers.

If F is the strength of the field, and ω the angle of the two

meridians, r_n and r_{n+1} the distance of two successive lines of force from the axis, we may take

$$F\frac{\omega}{2}(r_{n+1}^2 - r_n^2) = 1,$$

whence

$$r_n = \sqrt{\frac{2}{F\omega}}\sqrt{n}.$$

This method of representation has the inconvenience, as we see, of not representing a uniform field by equidistant lines of force.

140. Case of a Single Mass.—A single mass m gives a system of revolution about any axis passing through the acting mass. By planes perpendicular to an axis Ax (Fig. 31), the sphere may be divided into successive concentric zones of the same surface, and corresponding to the same flow.

The flow corresponding to the circular zone whose semi-angle

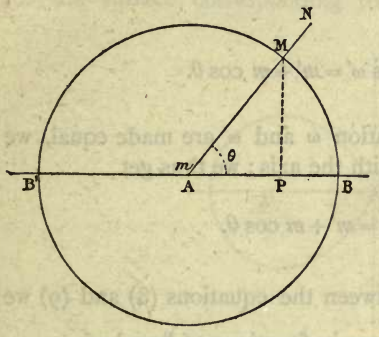

Fig. 31.

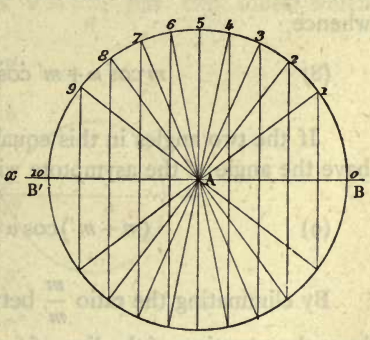

Fig. 32.

at the summit is θ, is proportioned to the cap of semi-aperture θ, that is to say, to the height PB or to $1 - \cos\theta$.

Let N be the order of the line of force AN, we shall have

$$\frac{N}{4\pi m} = \frac{1 - \cos\theta}{2},$$

whence

(7) $$\cos\theta = 1 - \frac{N}{2\pi m}.$$

To draw the lines of force, it is therefore sufficient to divide the diameters BB′ (Fig. 32) into $4\pi m$ parts, to draw the corresponding verticals, and to join the points of intersection with the circumference with the point A.

141. Any Two given Masses.—Let there now be two masses m and m' situate at A and A′ (Fig. 28); they form a system of revolution in reference to the straight line joining them. The total flow which traverses any zone of revolution whose semi-arc is PP′, is the sum of the flows which correspond to the angles ω and ω' for the two masses separately; that is to say $n+n'=N$.

For the same reason as above, the sheet which corresponds to the flow of the value N, and which passes through the point P is tangential at A to the cone whose angle is 2θ, which comprises the same flow for the mass m taken separately.

This sheet is also an asymptote to a cone, the apex of which is the centre of gravity O of the two masses.

The equation of the line of force AP, that is to say $n+n'=N$, gives

$$2\pi m(1-\cos\omega)+2\pi m'(1-\cos\omega')=N=2\pi m(1-\cos\theta),$$

whence

$$m\cos\omega+m'\cos\omega'=m'+m\cos\theta. \tag{8}$$

If the two angles in this equation ω and ω' are made equal, we have the angle of the asymptote with the axis; we thus get

$$(m+m')\cos\alpha=m'+m\cos\theta. \tag{9}$$

By eliminating the ratio $\frac{m}{m'}$ between the equations (8) and (9) we have the equation of the line of force in functions of θ and of α:

$$\frac{1-\cos\omega'}{\cos\omega-\cos\theta}=\frac{1-\cos\alpha}{\cos\alpha-\cos\theta}.$$

This is still a general method, and may be applied to any number of centres situate on the same right line.

The equation of a line of force starting from the mass m is

$$m\cos\omega+m'\cos\omega'+m''\cos\omega''\ldots\ldots=m'+m''+\ldots\ldots+m\cos\theta,$$

and that of the asymptote

$$(m+m'+m''\ldots\ldots)\cos\alpha=m'+m''+m'\ldots\ldots+m\cos\theta.$$

When the masses are all of the same sign all the lines of force are unlimited. If there are masses of contrary signs, the region which includes the finite lines of force emitted by positive masses, and absorbed by negative masses, is separated from the region which contains the unlimited lines of force by a bounding surface, the meridian section of which is determined by the value of the angle θ given by the preceding equation, in which a is made $= \pi$.

142. TWO EQUAL MASSES OF THE SAME SIGN.—If the system is made up of two equal masses of the same sign situate at A and A′, at the distance $2a$ (Fig. 33), the equipotential surfaces are given by the equation

$$V = \frac{m}{r} + \frac{m}{r'} = m\left(\frac{1}{r} + \frac{1}{r'}\right).$$

The meridian curves are lemniscates. The meridian curve for the surface corresponding to $V = \frac{2m}{a}$, has two lobes which

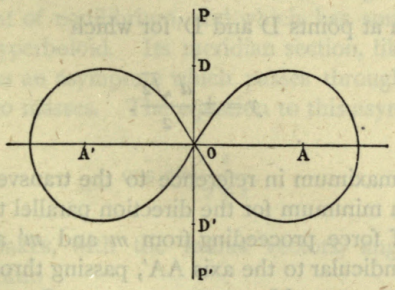

Fig. 33.

intersect at O. The point O is one of unstable equilibrium; the force there is equal to zero. At this point the potential has a minimum relative to the axis AA′, and a maximum in reference to the plane of symmetry PP′.

For all values of V higher than $\frac{2m}{a}$, the equipotential surface consists of two separate lobes the section of which has the form of an oval, and which surround each of the two centres. These ovals tend more and more to merge into circles as we approach the centre.

For lower values of V than $\frac{2m}{a}$, the surface consists of a single sheet the narrowing of which tends to disappear as V diminishes,

and which ultimately would be confounded at a great distance with a sphere the centre of which is the point O.

The equation of the lines of force is

$$\cos\omega + \cos\omega' = 1 + \cos\theta,$$

and that of the asymptote

$$2\cos\alpha = 1 + \cos\theta,$$

or

$$\cos\alpha = \cos^2\frac{1}{2}\theta.$$

The expression for the force at a point on the transverse axis OP is

$$F = \frac{2m}{r^2}\sin\omega = \frac{2my}{r^2}\frac{}{r} = 2m\frac{y}{(a^2+y^2)^{\frac{3}{2}}};$$

it is a maximum at points D and D′ for which

$$y = \pm\frac{a\sqrt{2}}{2}.$$

It is only a maximum in reference to the transverse axis, and on the contrary is a minimum for the direction parallel to AA′.

The lines of force proceeding from m and m' are separated by the plane perpendicular to the axis AA′, passing through the point O.

143. Two unequal Masses of the same Sign.—If two masses of the same sign are unequal, the general form of the equipotential surfaces is the same as in the preceding case excepting the symmetry. The point of equilibrium corresponding to the point of intersection of the surface with two sheets is defined by the ratio

$$\frac{m}{r^2} = \frac{m'}{r'^2}, \quad \text{or} \quad \frac{r}{r'} = \frac{\sqrt{m}}{\sqrt{m'}}.$$

Putting $r + r' = 2a$, we get from this

$$r = 2a\frac{\sqrt{m}}{\sqrt{m} + \sqrt{m'}} = 2a\frac{1}{1 + \sqrt{\frac{m'}{m}}}.$$

The equation of the lines of force is

$$m \cos \omega + m' \cos \omega' = m' + m \cos \theta.$$

They always form two distinct systems; the surface which separates them corresponds to $\theta = \pi$, and its equation is

$$m \cos \omega + m' \cos \omega' = m' - m.$$

If we put $\frac{m'}{m} = 1 + \epsilon$, the equation becomes

$$\cos \omega + (1 + \epsilon) \cos \omega' = \epsilon,$$

or in rectangular co-ordinates, the origin being taken in the middle of the distance $2a$,

$$\frac{x-a}{\sqrt{y^2+(x-a)^2}} + (1+\epsilon)\frac{x+a}{\sqrt{y^2+(x+a)^2}} = \epsilon.$$

This equation represents a surface of the sixth degree, which passes through the point of equilibrium, and which has some analogy with the sheet of a hyperboloid. Its meridian section, like all the other lines of force, has an asymptote which passes through the centre of gravity of the two masses. The equation to this asymptote is

$$\cos \alpha = \frac{\epsilon}{2+\epsilon}.$$

The force makes, with the radius vectors, angles β and β', defined by the ratio

$$(10) \qquad \frac{\sin \beta}{\sin \beta'} = \frac{\frac{m'}{r'^2}}{\frac{m}{r^2}} = \frac{m' r^2}{m r'^2} = \frac{m'}{m}\frac{\sin^2 \omega'}{\sin^2 \omega};$$

from which we get, for the value of the force,

$$F = \frac{m \cos \beta}{r^2} + \frac{m' \cos \beta'}{r'^2}.$$

144. TWO EQUAL MASSES OF OPPOSITE SIGNS. — We shall proceed to examine in greater detail the case of two equal masses of opposite signs, as it presents several important applications.

The equipotential surfaces whose equation is

$$V = m\left(\frac{1}{r} - \frac{1}{r'}\right),$$

are surfaces of an ovoidal form, with a single sheet, tending to merge into spheres in proportion as they approach the centres of action. All of them, which correspond to positive values of V, envelope the point A, while those which correspond to negative values envelope the point A'. They are separated by a symmetrical plane at zero potential.

The equation of the lines of force is

$$\cos\omega - \cos\omega' = 1 - \cos\theta = \frac{N}{2\pi m}. \tag{11}$$

These lines of force are all limited, proceeding from the point A, and terminating at the point A'; they are evidently symmetrical in reference to the plane of zero potential, which is perpendicular to the axis AA' at 0, the middle of the distance AA' (Fig. 34).

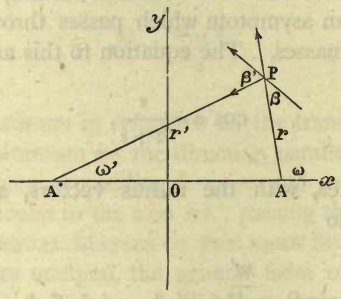

Fig. 34.

145. The angles β and β' which the force makes with the radius vectors are still determined by the equation (10), which gives

$$\frac{\sin\beta}{r^2} = \frac{\sin\beta'}{r'^2}. \tag{12}$$

146. The expression of the force is

$$F = m\left[\frac{\cos\beta}{r^2} + \frac{\cos\beta'}{r'^2}\right]. \tag{13}$$

Its value at P_1 on the axis AA′, at a distance d from the centre, is

$$(14) \qquad F_1 = m\left[\frac{1}{(d-a)^2} - \frac{1}{(d+a)^2}\right] = 2ma\,\frac{2d}{(d^2-a^2)^2}$$

and on the transverse axis at the same distance d from the centre, or at the distance ρ from either of the masses,

$$(15) \qquad F_2 = 2ma\frac{1}{\rho^3} = 2ma\,\frac{1}{(d^2+a^2)^{\frac{3}{2}}}.$$

The product $2ma$, of one of the masses by the distance separating them, which is called the *magnetic moment* in the corresponding problem in magnetism, may be called the *electrical moment* of the system.

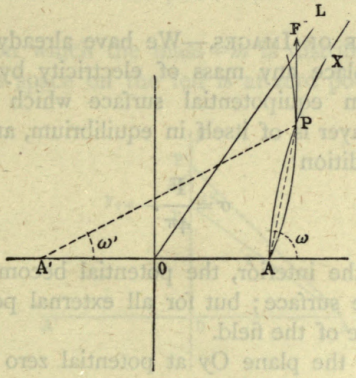

Fig. 35.

147. When the force is perpendicular to the axis

$$\beta = \frac{\pi}{2} - \omega, \qquad \sin\beta = \cos\omega,$$

$$\beta' = \frac{\pi}{2} + \omega', \qquad \sin\beta' = \cos\omega';$$

equation (12) becomes

$$\frac{\cos\omega}{r^2} = \frac{\cos\omega'}{r'^2}.$$

This is the equation of the curve APX (Fig. 35) which passes through all the points of the plane where the force is vertical. It

consists of two symmetrical branches, starting from A and A′ tangentially to the vertical, and which are asymptotes to a straight line OL.

In order to determine the direction of the asymptote, let us consider a very distant point; then, if δ is the very small difference $\omega - \omega'$, and observing that the angles ω and ω' ultimately become equal,

$$\frac{\cos\omega}{r^2} = \frac{\cos\omega'}{r'^2} = \frac{\cos\omega' - \cos\omega}{r'^2 - r^2} = \frac{\sin\omega.\delta}{2r(r'-r)} = \frac{\sin\omega}{2r^2}\cdot\frac{r\delta}{r'-r}$$

$$= \frac{\sin\omega.2a\sin\omega}{2r^2.2a\cos\omega} = \frac{1}{2r^2}\frac{\sin^2\omega}{\cos\omega},$$

whence

$$\tan^2\omega = 2.$$

148. Principle of Images.—We have already seen (59) that we can always replace any mass of electricity by an equal mass distributed over an equipotential surface which completely surrounds it. This layer is of itself in equilibrium, and its density is defined by the condition

$$\sigma = \frac{F}{4\pi}.$$

For all points in the interior, the potential becomes constant and equal to that of the surface; but for all external points, nothing is changed in the state of the field.

Let us consider the plane Oy at potential zero (Fig. 34) in the preceding problem. For all points on the right we may replace the mass $-m$ on A′ by an equal mass in equilibrium on the plane. The density will be P_2 at each point, the force F_2 being directed towards the left,

$$-\sigma = \frac{F_2}{4\pi} = \frac{(2ma)}{4\pi}\cdot\frac{1}{\rho^3}.$$

We see thus that it is *inversely as the cube* of the distance of the point in question P_2 from the point A.

It may be observed, from this law of distribution, that the charge of an element of the plane is everywhere proportional to the angle which it subtends at the point A. In fact, the charge of a surface element dS (Fig. 36) is

$$-\sigma d\mathrm{S} = \frac{m}{2\pi}\cdot\frac{1}{\rho^2}\cdot\frac{a}{\rho}d\mathrm{S}.$$

Now, $\frac{a}{\rho} d\mathrm{S}$ is the projection $d\mathrm{S}_1$ of the element $d\mathrm{S}$ on a plane perpendicular to ρ, and $\frac{d\mathrm{S}_1}{\rho^2}$ is the angle $d\theta$ under which the element $d\mathrm{S}$ is seen from the point A.

We have then

$$-\sigma d\mathrm{S} = \frac{m}{2\pi} d\theta.$$

If the plane, or the mass $-m$, did not exist, the flow of force from the mass m in the angle $d\theta$ would be $md\theta$. In the present case the flow received by the surface $d\mathrm{S}$ is $4\pi \frac{m}{2\pi} d\theta = 2md\theta$. The distribution of this flow is the same as if the mass m were alone there, but the flow is doubled at each point since all the lines of force meet the plane.

The plane, on which the mass $-m$ is distributed, being at zero potential, all the space on the left is at zero potential. This is the

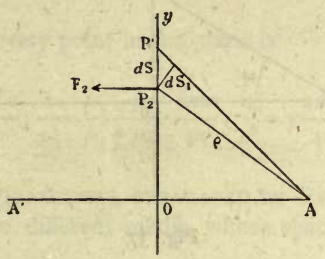

Fig. 36.

case of an unlimited conducting plane Oy, in connection with the earth, and under the influence of a mass of electricity $+m$ placed at a point A. Such a plane completely intercepts the action of the mass m on points behind it; it plays the part of an *electrical screen.*

Thus, the mass $+m$ being placed at A, in the presence of a conducting plane Oy in connection with the earth, this plane may be replaced, for all points on the right, by a mass $-m$ at the point A′ symmetrical with A.

Sir W. Thomson looks upon the mass $-m$ at A′, considered in reference to the plane Oy in connection with the earth, as the *image* of the mass $+m$ at A. The analogy between the electrical phenomenon and the corresponding optical problem is at once evident. If the point A is a source of light and the plane Oy a reflecting

mirror, the image of A is a virtual one, and is formed at A′; the illumination of the space on the right of the plane is the same as if this plane were replaced by a source of light placed at A′, and the intensity of this virtual source would be equal to A, if the reflecting power of the plane were equal to unity.

149. INDUCTION IN A MEDIUM CONSISTING OF TWO DIELECTRICS SEPARATED BY A PLANE.—The principle of images enables us to determine the condition of two unlimited dielectrics, separated by a plane surface, in one of which is the acting mass.

Let m be this mass placed at the point A (Fig. 37), μ_1 and μ_2 the inductive powers of two dielectrics separated by the plane Q, the acting mass being situate in the former.

Equilibrium may be established by imagining that on the plane Q, a layer m' is distributed as it would be on an uninsulated conducting

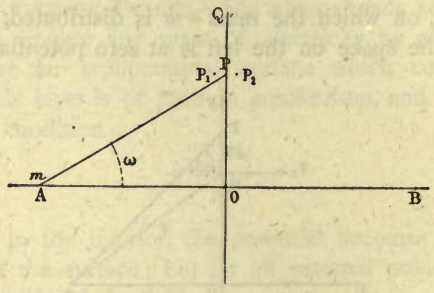

Fig. 37

plane under the influence of a mass m', placed at A, or at the symmetrical point B; in other words, the plane would act on all points on its left like a mass m' placed at A, and on all points on the right like the same mass m' placed at B. The potential near the point P, taken in the plane Q, is, in the first medium, at P_1,

$$V_1 = \frac{m}{PA} + \frac{m'}{PB},$$

and in the second medium, at P_2,

$$V_2 = \frac{m}{PA} + \frac{m'}{PA}.$$

The perpendicular components of the force are, at the same points,

$$N_1 = \frac{\cos \omega}{\rho^2}(m - m'),$$

$$N_2 = \frac{\cos \omega}{\rho^2}(m + m').$$

In order to satisfy the equation of continuity of the dielectrics, the product of the perpendicular component by the specific inductive capacity, must be the same on both sides of the surface of separation, which gives

$$\mu_1(m - m') = \mu_2(m + m'),$$

whence

$$m' = m\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} = m\gamma.$$

The density at every point in the plane is

$$\sigma' = \frac{2m'a}{4\pi} \cdot \frac{1}{\rho^3} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{2ma}{4\pi} \cdot \frac{1}{\rho^3} = \gamma \frac{2ma}{4\pi} \cdot \frac{1}{\rho^3}.$$

150. THREE DIELECTRICS SEPARATED BY PARALLEL PLANES.— Let us imagine three different media, whose specific inductive capa-

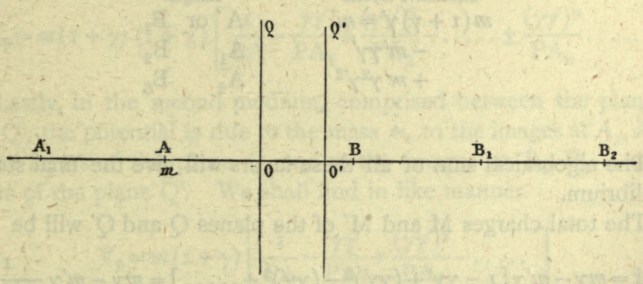

Fig. 38.

cities are respectively equal to μ_1, μ_2 and μ_3, separated by parallel planes Q and Q', and let the acting mass m be situate in the first medium, at A (Fig. 38).

Let us further take

$$\gamma = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

$$\gamma' = \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3}.$$

The condition of equilibrium on the plane Q is satisfied by a layer $m\gamma$, which acts on each side as if it were at A or at B.

The mass m at A, and the layer $m\gamma$ of the plane Q, produce on the plane Q' a layer $m\,(1+\gamma)\gamma' = m'$, which will act as if it were concentrated at A or at B_1.

The layer m' reacting on the plane Q, will produce there a layer $-m'\gamma$, the image of which is at B_1 or at A_1.

In like manner the layer $-m'\gamma$ at Q gives on Q' a layer $-m'\gamma\gamma'$ the image of which is A_1 or B_2..., etc.

The determination of these successive layers is nothing but the application of Murphy's method.

We shall thus have, from layer to layer:
on the plane Q,

Successive layers.	Images.
$m\gamma$	A or B
$-m'\gamma$	A_1 B_1
$+m'\gamma^2\gamma'$	A_2 B_2
$-m'\gamma^3\gamma'^2$	A_3 B_3
⋮	⋮ ⋮

and on the plane Q',

Successive layers.	Images.
$m(1+\gamma)\gamma' = m'$	A or B_1
$-m'\gamma\gamma'$	A_1 B_2
$+m'\gamma^2\gamma'^2$	A_2 B_3
⋮	⋮ ⋮

The algebraical sum of all these layers will give the final state of equilibrium.

The total charges M and M' of the planes Q and Q' will be

$$M = m\gamma - m'\gamma\left[1 - \gamma\gamma' + (\gamma\gamma')^2 - (\gamma\gamma')^3 + \ldots\ldots\right] = m\gamma - m'\gamma\frac{1}{1+\gamma\gamma'}$$

$$= m\gamma\left[1 - \frac{(1+\gamma)\gamma'}{1+\gamma\gamma'}\right] = m\frac{\gamma(1-\gamma')}{1+\gamma\gamma'}.$$

$$M' = m'\left[1 - \gamma\gamma' + (\gamma\gamma')^2 - \ldots\ .\right] = m'\frac{1}{1+\gamma\gamma'} = m\frac{\gamma'(1+\gamma)}{1+\gamma\gamma'}.$$

The density at every point is equal to the algebraical sum of the densities of all the layers superposed.

The potential V, at a point P in the former medium, may be considered as produced by the mass m, and all the images situate at B, B_1, B_2..., etc.

At the different points B_1, B_2..., there are two different images arising from the layers on the two planes Q and Q', and we have:

at B_1

$$m' - m'\gamma = m'(1-\gamma) = m(1-\gamma^2)\gamma',$$

at B_2

$$-m'\gamma\gamma' + m'\gamma^2\gamma' = -m'\gamma\gamma'(1-\gamma) = -m(1-\gamma^2)\gamma'.\gamma\gamma',$$

.....

at B_{n+1}

$$\pm m'(\gamma\gamma')^n \pm m'(\gamma\gamma')^n\gamma = \pm m'(\gamma\gamma')^n(1-\gamma) = \pm m(1-\gamma^2)\gamma'(\gamma\gamma')^n;$$

which gives

$$V_1 = m\left\{\frac{1}{PA} + \frac{\gamma}{PB} + (1-\gamma^2)\gamma'\left[\frac{1}{PB_1} - \frac{\gamma\gamma'}{PB_2} + \frac{(\gamma\gamma')^2}{PB_3} \ldots \pm \frac{(\gamma\gamma')^n}{PB_{n+1}} \pm \ldots\right]\right\}$$

The potential V_3 in the third medium is produced, in like manner, by the images situate at points A, A_1 A_2..., on which the masses are:

$$m + m\gamma + m' = m(1+\gamma)(1+\gamma'),$$
$$-m'\gamma - m'\gamma\gamma' = -m(1+\gamma)(1+\gamma')\gamma\gamma',$$
.....
$$\pm(m'\gamma^n\gamma'^{n-1} + m'\gamma^n\gamma'^n) = \pm m(1+\gamma)(1+\gamma')(\gamma\gamma')^n;$$

we have then

$$V_3 = m(1+\gamma)(1+\gamma')\left[\frac{1}{PA} - \frac{\gamma\gamma'}{PA_1} + \frac{(\gamma\gamma')^2}{PA_2} - \ldots\ldots \pm \frac{(\gamma\gamma')^n}{PA_n} \ldots\ldots\right].$$

Lastly, in the second medium, comprised between the planes Q and Q', the potential is due to the mass m, to the images at A_1, A_2... of the layers of the plane Q, and to the images at B_1, B_2... of the layers of the plane Q'. We shall find in like manner

$$V_2 = m(1+\gamma)\left[\frac{1}{PA} - \frac{\gamma\gamma'}{PA_1} + \frac{(\gamma\gamma')^2}{PA_2} - \ldots\ldots\right]$$
$$+ m(1+\gamma)\gamma'\left[\frac{1}{PB_1} - \frac{\gamma\gamma'}{PB_2} + \frac{(\gamma\gamma')^2}{PB_3} - \ldots\ldots\right].$$

If the third medium is identical with the first, we simply put $\mu_1 = \mu_3$, and then we get $\gamma' = -\gamma$.

We thus obtain

$$M = m\frac{\gamma(1+\gamma)}{1-\gamma^2} = m\frac{\gamma}{1-\gamma},$$

$$M' = -m\frac{\gamma(1+\gamma)}{1-\gamma^2} = -m\frac{\gamma}{1-\gamma},$$

$$V_1 = m\left\{\frac{1}{PA} + \frac{\gamma}{PB} - (1-\gamma^2)\gamma\left[\frac{1}{PB_1} + \frac{\gamma^2}{PB_2} + \ldots + \frac{\gamma^{2n}}{PB_{n+1}} + \ldots\right]\right\}$$

$$V_3 = m(1-\gamma^2)\left[\frac{1}{PA} + \frac{\gamma^2}{PA_1} + \frac{\gamma^4}{PA_2} + \ldots\ldots + \frac{\gamma^{2n}}{PA_n} + \ldots\ldots\right],$$

$$V_2 = m(1+\gamma)\left\{\left[\frac{1}{PA} + \frac{\gamma^2}{PA_1} + \ldots\ldots\right] - \gamma\left[\frac{1}{PB_1} + \frac{\gamma^2}{PB_2} + \ldots\ldots\right]\right\}.$$

Denoting by α and β the two series containing the distances PA, $PA_1 \ldots$, P B_1, $PB_2 \ldots$, which are determinate functions of the co-ordinates of the point P, we have simply

$$V_1 = m\left[\frac{1}{PA} + \frac{\gamma}{PB} - (1-\gamma^2)\gamma\beta\right],$$

$$V_3 = m(1-\gamma^2)\alpha,$$

$$V_2 = m(1+\gamma)[\alpha - \gamma\beta].$$

151. Two Equal Masses of Opposite Signs infinitely near each other.—Let us suppose that the two equal masses of opposite signs $+m$ and $-m$ of problem (144) are infinitely near—or, what

Fig. 39.

amounts to the same thing, let us consider the condition of the field at a distance which is very great compared with the distance $2a$ of the two points A and A'. The value of the potential at a point P (Fig. 39)

$$V = m\left(\frac{1}{r} - \frac{1}{r'}\right) = m\frac{r'-r}{rr'},$$

reduces to

$$V = m\frac{r' - r}{r^2} = 2am\frac{\cos\omega}{R^2} = 2am\frac{x}{R^3},$$

ω being the angle of the direction OP with the axis A′A.

Let ϖ be a surface, a circle, for instance, traced by the point O perpendicularly to AA′, and let θ be the solid angle under which this surface is seen from the point P; we have

$$R^2\theta = \varpi\cos\omega,$$

and therefore

$$V = \frac{2ma}{\varpi}\theta.$$

taking $\varpi = 2ma$, we get

$$V = \theta. \tag{16}$$

Thus, *the value of the potential at a point, is the solid angle under which we see from this point, a surface equal to the electrical moment 2ma of the two masses and perpendicular to the middle of the straight line joining them.*

The equation of the equipotential surfaces,

$$V = \frac{\varpi\cos\omega}{R^2} = \frac{\varpi x}{R^3}, \tag{17}$$

shows that all these surfaces are similar, and that for the same direction ω, the values of R are inversely as the square roots of the potentials.

152. In the equation of the lines of force,

$$\cos\omega' - \cos\omega = \frac{N}{2\pi m},$$

the first member may be transformed in the following manner, denoting by δ the infinitely small difference $\omega - \omega'$:

$$\cos\omega' - \cos\omega = d.\cos\omega = \sin\omega.\delta = \sin\omega.\frac{2a\sin\omega}{R} = \frac{2a}{R}\sin^2\omega.$$

we have then

$$\frac{\sin^2\omega}{R} = \frac{1}{2a}.\frac{N}{2\pi m} = \frac{N}{2\pi(2ma)} = \frac{N}{2\pi\varpi}. \tag{18}$$

All these curves are similar, and for the same direction, R is inversely as N. They are tangential to the axis at the origin; the

loci of the points where the tangent is vertical is evidently the asymptote found in the preceding problem (144), and the equation of which is

$$\tan^2 \omega = 2.$$

153. Equations (14) and (15) of (146) give for the values of the force on the axis, and on the transversal, at A and B (Fig. 40),

$$F_1 = 2\frac{\varpi}{R^3},$$

$$F_2 = \frac{\varpi}{R^3},$$

and therefore

$$F_1 = 2F_2.$$

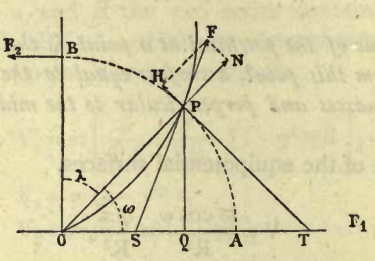

Fig. 40.

From equation (17) we have for a point P at the same distance in any direction whatever ω,

$$(19)\qquad \begin{aligned} X &= -\frac{\partial V}{\partial x} = -\varpi\left(\frac{1}{r^3} - 3\frac{x^2}{r^2}\right) = \frac{\varpi}{R^3}(3\cos^2\omega - 1), \\ Y &= -\frac{\partial V}{\partial y} = -\varpi\frac{-3xy}{r^5} = \frac{\varpi}{R^3}3\sin\omega\cos\omega. \end{aligned}$$

Through the point O as centre, draw a circle of radius R, passing through the point P, and consider at this point the perpendicular component F_n, and the tangential component F_t of the force; we have

$$(20)\qquad \begin{aligned} F_n &= X\cos\omega + Y\sin\omega = \frac{\varpi}{R^3}2\cos\omega, \\ F_t &= -X\sin\omega + Y\cos\omega = \frac{\varpi}{R^3}\sin\omega. \end{aligned}$$

If i be the inclination of the force F with the tangent, and λ the complement of the angle ω, we have

$$\tan i = \frac{F_n}{F_t} = 2 \cot \omega = 2 \tan \lambda. \tag{21}$$

We get lastly, for the force itself,

$$\begin{aligned} F^2 = X^2 + Y^2 = F_n^2 + F_t^2 &= \left(\frac{\varpi}{R^3}\right)^2 (3 \cos^2 \omega + 1) \\ &= \left(\frac{\varpi}{R^3}\right)^2 (3 \sin^2 \lambda + 1) \end{aligned} \tag{22}$$

154. Prolong the tangent as far as the axis at T, and the direction of the force to S; the triangles OPS and OPT give

$$\frac{PS}{\sin \omega} = \frac{OS}{\cos i},$$

$$\frac{PS}{\cos \omega} = \frac{ST}{\sin i};$$

from which we have

$$OS \frac{\sin \omega}{\cos i} = ST \frac{\cos \omega}{\sin i},$$

and finally

$$OS \tan i = ST \cot \omega = ST \tan \lambda.$$

As $\tan i = 2 \tan \lambda$, we see that

$$ST = 2OS,$$

whence

$$OT = 3OS. \tag{23}$$

This theorem is due to Gauss.

The value of the force is easily expressed as a function of the same lines. We have, in fact,

$$R = OT \cos \omega = 3OS \cos \omega,$$

$$OQ = R \cos \omega,$$

and, consequently,

$$3 \cos^2 \omega = \frac{OQ}{OS}.$$

We obtain then, by substitution,

$$F^2 = \left(\frac{\varpi}{R^3}\right)\left(\frac{OQ}{OS} + 1\right). \tag{24}$$

155. In valuing the force at each point, the electrical masses only affect the result by their moment $2ma = \varpi$, which may remain finite for suitable values of m, although the distance $2a$ is infinitely small. The total flow of force proceeding from the two infinitely near centres is not therefore determined, but the flow from a sphere of given radius R may be easily calculated.

From equation (20) the value of the perpendicular component at point P, corresponding to the angle ω, is

$$F_n = \frac{\varpi}{R^3} 2 \cos \omega.$$

The surface of the bow whose angular aperture is 2ω, being equal to $2\pi R^2(1 - \cos \omega)$, that of the elementary zone corresponding to the angle $d\omega$ is

$$dS = 2\pi R^2 \sin \omega d\omega.$$

The flow of force which traverses this zone is therefore

$$dQ = \frac{4\pi\varpi}{R} \sin \omega \cos \omega \, d\omega,$$

and the total flow corresponding to the angle ω is

$$Q_\omega = \frac{4\pi\varpi}{R} \int \sin \omega \cos \omega \, d\omega = \frac{2\pi\varpi}{R} \sin^2 \omega.$$

This expression is nothing more than that of the line of force, of the order N, which terminates at the contour of the zone in question, and it might have been written directly.

If ω be made equal to $\frac{\pi}{2}$, we shall have the total flow on one side of the transverse plane OB; the value of this flow is $\frac{2\pi\varpi}{R}$; it is seen to be inversely as R.

To trace on a meridian plane the lines of force which correspond

to the flows represented by the numbers 1, 2, 3, 4..., we need only take, on the transverse axis, lengths corresponding to the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$....., and by equation (18) draw the lines of force which cut the axis in these different points.

156. Induction on an Infinitely Small Body.—The system of two equal masses of opposite signs infinitely near each other, represents the condition of an infinitely small body, conductor or not, originally in the neutral state, and placed in any given electrical field. The body is in effect covered with two layers of equal masses, and of opposite signs, each of which acts as if it were concentrated in its centre of gravity.

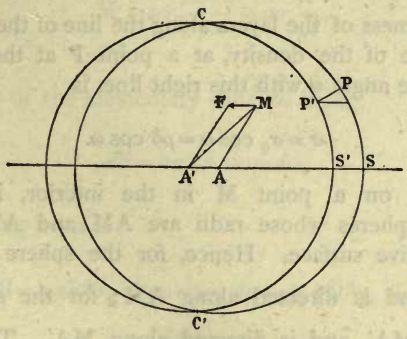

Fig. 41.

This is also the case with any body, originally in the neutral state, (that is with a total charge null,) when its action at a great distance is considered.

157. Polarized Sphere.—Layers of Gliding.*—Let us consider two spheres S and S′ of the same radius (Fig. 41), of uniform densities $+\rho$ and $-\rho$, and whose centres A and A′ are at an infinitely small distance δ. This system is in fact equivalent to that of two equal layers of opposite signs distributed on the two halves of a spherical surface.

This particular form of electrification is of great interest, and corresponds in magnetism to a very simple method of magnetisation. For the sake of brevity we may apply the term *layers of gliding*

* What are here spoken of as layers of gliding (*couches de glissement*), are the result of a purely fictitious geometrical operation, which does not aim at representing a real phenomenon, or a particular constitution of the electrified body. We shall retain the expression electrical displacement to denote the mechanical modification of the medium which Maxwell had in view in his theory of dielectrics.

to those which are thus produced by two homogeneous masses equal in density and of opposite signs, one of which has moved through an infinitely small distance.

The medium may be considered to be polarized, and the axis of electrical polarization is parallel to the direction along which the displacement has taken place.

In the present case the density of the layer at each point is proportional to the corresponding thickness P'P of the part which is not common to the two spheres. Denoting by σ_0 this density on the line of the centres, we shall have

$$\sigma_0 = \rho\delta.$$

As the thickness of the layers along the line of the centres is constant, the value of the density, at a point P at the end of radius which makes the angle ω with this right line, is

$$\sigma = \sigma_0 \cos\omega = \rho\delta \cos\omega.$$

The action on a point M in the interior, is that of two homogeneous spheres whose radii are AM and A'M, on a point of their respective surface. Hence, for the sphere A, it is equal to $\frac{4}{3}\pi\rho.\mathrm{AM}$, and is directed along AM; for the sphere A', it is equal to $\frac{4}{3}\pi\rho.\mathrm{MA'}$, and is directed along MA'. The resultant is therefore proportional to AA' and has the value

$$\frac{4}{3}\pi\rho.\,\mathrm{AA'} = \frac{4}{3}\pi\rho\delta = \frac{4}{3}\pi\sigma_0;$$

it is constant. Let us denote this force by F_i, and reckon it positively from left to right, we shall have

$$\mathrm{F}_i = -\frac{4}{3}\pi\sigma_0.$$

In the interior of the sphere, the equipotential surfaces are planes perpendicular to the axis AA' and are equidistant; the potential at a point varies proportionally to the abscissa x of the point, and as it is zero at the centre, we have

$$\mathrm{V}_i = \frac{4}{3}\pi\sigma_0 x.$$

For the outside, the layer in question may be replaced by two homogeneous spheres, or by two masses of opposite signs equal to $\frac{4}{3}\pi R^3\rho$ concentrated at A and A′, and the moment of which is

$$\varpi = \delta . \frac{4}{3}\pi R^3\rho = \frac{4}{3}\pi R^3\sigma_0 = u\sigma_0,$$

u being the volume of the sphere. We shall have then, for the distance r, in a direction at an angle ω with the axis,

$$V_e = u\sigma_0 \frac{\cos\omega}{r^2} = u\sigma_0 \frac{x}{r^3}.$$

The surface of the elementary zone $d\omega$ being

$$dS = 2\pi R^2 \sin\omega\, d\omega,$$

the corresponding mass is

$$\sigma dS = 2\pi R^2\sigma_0 \sin\omega\cos\omega\, d\omega = d . \pi R^2\sigma_0 \sin^2\omega.$$

The total mass M of each of the layers is then

$$M = \int \sigma dS = \int_0^{\frac{\pi}{2}} d . \pi R^2\sigma_0 \sin^2\omega = \pi R^2\sigma_0.$$

This result might have been directly obtained by considering that at each point of the layer the thickness along the axis being equal to δ, the total volume is equal to the product of this constant thickness by the projection of the hemisphere on a plane perpendicular to AA′.

The flow of force from the positive layer is

$$Q = 4\pi M = (4\pi R^2)\pi\sigma_0 = S\pi\sigma_0,$$

S being the whole surface of the sphere.

158. It will be useful to collate here all the preceding results, and to express each of them by quantities as a function of the

maximum density δ_0, or of the internal force F_i; if a be the radius of the sphere and u its volume, we have

$$F_i = -\frac{4}{3}\pi\rho\delta = -\frac{4}{3}\pi\sigma_0 = -\frac{u\sigma_0}{a^3},$$

$$\varpi = u\sigma_0 = -a^3 F_i,$$

$$V_i = \frac{u\sigma_0 x}{a^3} = -F_i x = -F_i r\cos\omega,$$

$$V_e = u\sigma_0 \frac{x}{r^3} = -F_i \frac{a^3}{r^3} x = -F_i \frac{a^3\cos\omega}{r^2},$$

$$X_e = \frac{u\sigma_0}{r^3}(3\cos^2\omega - 1) = F_i \frac{a^3}{r^3}(1 - 3\cos^2\omega),$$

$$Y_e = \frac{u\sigma_0}{r^3} 3\sin\omega\cos\omega = -F_i \frac{a^3}{r^3} 3\sin\omega\cos\omega,$$

$$F_n = \frac{u\sigma_0}{r^3} 2\cos\omega = -2F_i \frac{a^3}{r^3}\cos\omega,$$

$$F_t = \frac{u\sigma_0}{r^3}\sin\omega = -F_i \frac{a^3}{r^3}\sin\omega,$$

$$M = \pi a^2\sigma_0 = \frac{3}{4}\frac{u\sigma_0}{a} = -\frac{3}{4}F_i a^2,$$

$$Q = S\pi\sigma_0 = -S\frac{3}{4}F_i = -3\pi a^2 F_i.$$

159. CONDUCTING SPHERE IN A UNIFORM FIELD.—Suppose now that a sphere thus electrified is placed in a uniform field, of strength ϕ, parallel to the axis of x, and let V_0 be the value of the potential in the plane which passes through the centre. If we have

$$F_i + \phi = 0,$$

the resultant force is null in the whole sphere; the potential is therefore constant, and there is equilibrium if the sphere is a conductor.

To obtain the electrical state of a conducting sphere in a uniform field of strength ϕ, F_i may be replaced by $-\phi$ in all the preceding formulæ, which gives

$$\sigma_0 = \frac{3}{4\pi}\phi,$$

$$\varpi = a^3\phi,$$

$$V_i = \phi r \cos\omega,$$

$$V_e = \phi\frac{a^3}{r^2}\cos\omega,$$

$$X_e = \phi\frac{a^3}{r^3}(3\cos^2\omega - 1),$$

$$Y_e = \phi\frac{a^3}{r^3}3\sin\omega\cos\omega,$$

$$F_n = 2\phi\frac{a^3}{r^3}\cos\omega,$$

$$F_t = \phi\frac{a^3}{r^3}\sin\omega,$$

$$M = \frac{3}{4}a^2\phi,$$

$$Q = 3\pi a^2\phi.$$

160. If R be the radius of the circle, drawn on an equipotential surface, through which would pass the same flow in the original field—that is to say, of the circle which comprises all the lines of force directed towards the conducting sphere—we have

$$\pi R^2\phi = 4\pi M = 3\pi a^2\phi,$$
$$R^2 = 3a^2.$$

This circle has therefore a surface three times as great as a great circle of the sphere.

All the lines of force terminate perpendicularly at the surface of the sphere and proceed perpendicularly from it, always excepting those which fall upon the equator; these make with the normal an angle of 45°.

In fact, for any point at a distance r in the direction ω, the angle θ of the resulting force with the radius vector r, is given by the ratio

$$\tan\theta = \frac{\phi\sin\omega - F_t}{\phi\cos\omega + F_n} = \tan\omega\,\frac{1-\frac{a^3}{r^3}}{1+2\frac{a^3}{r^3}}.$$

This angle θ is always null when $r=a$—that is, when the point is on the sphere. For all points on the equator, however, the angle ω is equal to $\frac{\pi}{2}$, and the expression assumes an indeterminate form.

Let us suppose that the angle ω is very little different from $\frac{\pi}{2}$, the angle θ for a point P (Fig. 42) near the surface is

$$\tan\theta = \frac{CQ}{QP} = \frac{a\left(\frac{\pi}{2}-\omega\right)}{r-a} = \frac{a\tan\left(\frac{\pi}{2}-\omega\right)}{r-a} = \frac{a}{r-a}\cdot\frac{1}{\tan\omega}.$$

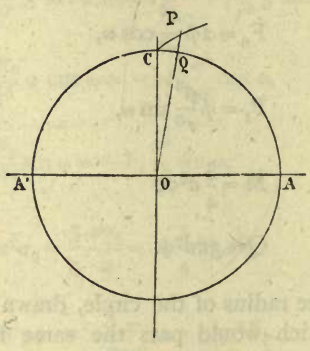

Fig. 42.

Multiplying this equation by the preceding, which always holds, and observing that the difference $r-a$ is very small, we get

$$\tan^2\theta = \frac{a}{r-a}\cdot\frac{r^3-a^3}{r^3+2a^3} = \frac{a}{r-a}\cdot\frac{3a^2(r-a)}{r^3+2a^3} = \frac{3a^3}{r^3+2a^3} = 1.$$

The lines of force which touch the sphere on the equator make, therefore, with the surface, an angle of 45°.

The equipotential surface at the original potential V_0 of the centre of the sphere is a plane which terminates at the equator, and is thus prolonged by the surface of the sphere itself. The equator is a line of equilibrium.

161. UNINSULATED CONDUCTING SPHERE IN A UNIFORM FIELD.—It is easy to pass from the case we have been treating to that of a sphere situate in a uniform field, and in connection with the earth. For this, the internal potential which had the constant value V_0 must be null; this condition is fulfilled by superposing on the preceding condition a uniform layer capable of producing in the interior a potential equal and of opposite sign to V_0. Let $-M'$ be the mass of this layer and $-\sigma'$ its density, we shall have

$$V_0 = \frac{M'}{a} = \frac{4\pi a^2 \sigma'}{a} = 4\pi a \sigma',$$

whence

$$\sigma' = \frac{V_0}{4\pi a}.$$

The resultant density at any point will be

$$\sigma = \sigma_0 \cos\omega - \sigma' = \frac{3}{4\pi}\phi \cos\omega - \frac{V_0}{4\pi a},$$

$$4\pi\sigma = 3\phi \cos\omega - \frac{V_0}{a}.$$

For the density to be zero at A, we must have

$$3\phi a = V_0$$

at the pole of the sphere.

If V_1 and V_2 are the original potentials at A′ and A, the strength of the field is

$$\phi = \frac{V_2 - V_1}{2a},$$

and the potential at the centre

$$V_0 = \frac{V_1 + V_2}{2}.$$

The preceding condition reduces to

$$\frac{3}{2}(V_2 - V_1) = \frac{V_1 + V_2}{2},$$

whence

$$V_2 = 2V_1.$$

For the density to be null at A, the original values of the potential at the two poles of the sphere must be in the ratio of 1 to 2.

The density at any other point is negative, and therefore the surface of the sphere is entirely negative as long as $V_2 < 2V_1$.

In the contrary case, the greater part of the surface is still negative, but about the point A there is a more or less extensive zone of positive electricity.

162. DIELECTRIC SPHERE IN A UNIFORM FIELD.—Uniform polarization, or electrification by layers of displacement, also represents, on Poisson's theory, the condition of a dielectric in a uniform field. Yet if ϕ be the strength of the field, F_i the internal force due to the fictive layer, the resultant force at each point of the interior, instead of being zero, will have a constant value equal to $\phi + F_i$.

We can demonstrate that the condition relative to the equilibrium of dielectrics is then satisfied—that is to say, that there is a constant ratio over the whole surface between the perpendicular components on the interior and on the exterior.

This ratio μ, being given by the nature of the dielectric, will enable us to determine the force F_i, and consequently the distribution of the fictive layer.

For a point P on the surface in a direction ω, the external perpendicular component is $\phi \cos\omega + F_n = (\phi - 2F_i)\cos\omega$, and the internal perpendicular component $(\phi + F_i)\cos\omega$. The ratio of these two forces

$$\mu = \frac{(\phi - 2F_i)\cos\omega}{(\phi + F_i)\cos\omega} = \frac{\phi - 2F_i}{\phi + F_i}$$

is therefore constant, and we deduce from it

$$F_i = -\frac{\mu - 1}{\mu + 2}\phi.$$

It thus appears that the problem is completely determinate, and that the state of the sphere is identical with that of a conducting sphere of the same radius situate in a uniform field, the strength of which is $\phi\frac{\mu - 1}{\mu + 2}$. We deduce from this (159)

$$\sigma_0 = \frac{3}{4\pi}\frac{\mu - 1}{\mu + 2}\phi.$$

163. The flow of force from the sphere is equal to the flow which traverses it, $\pi a^2 (\phi + F_i)$ increased by the flow $-3\pi a^2 F_i$, which corresponds to each of the surface layers.

We have therefore

$$Q = \pi a^2 (\phi + F_i - 3F_i) = \pi a^2 (\phi - 2F_i)$$

$$= \pi a^2 \phi \left(1 + 2\frac{\mu - 1}{\mu + 2}\right) = 3\pi a^2 \phi \frac{\mu}{\mu + 2}.$$

The equivalent circle of flow on the original equipotential surfaces would have a radius determined by the equation

$$R^2 = 3a^2 \frac{\mu}{\mu + 2} = a^2 \frac{3}{1 + \frac{2}{\mu}};$$

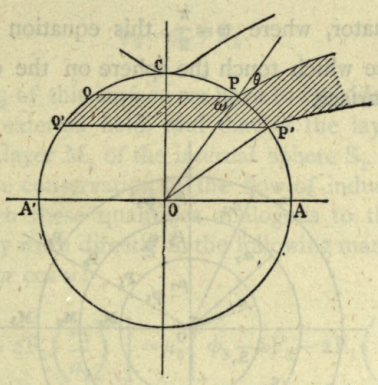

Fig. 43.

so long as $\mu > 1$ this radius is always greater than that of the sphere. The force in the interior is

$$\phi + F_i = \phi \left(1 - \frac{\mu - 1}{\mu + 2}\right) = \frac{3}{2 + \mu} \phi;$$

it is equal to $\frac{3}{4}\phi$ if $\mu = 2$, which is approximately the case with most dielectrics, and becomes equal to $\frac{3\phi}{\mu}$ when μ is very large.

Near the pole A, on the outside, the force is

$$\phi - 2F_i = \phi \frac{3\mu}{\mu + 2};$$

it is equal to $\frac{3}{2}\phi$ for $\mu = 2$, and becomes 3ϕ when μ is very great. This force, therefore, is then thrice its primitive value; this is the case with conductors.

In the present case the external lines of force are no longer perpendicular to the surface. It could be easily shown that the tangential components are equal, and that the ratio of the angles θ and ω (Fig. 43) of the perpendicular to the lines of force on the outside and inside satisfies the law of refraction.

$$\tan\theta = \frac{\tan\omega}{\mu}.$$

For the equator, where $\omega = \frac{\pi}{2}$, this equation also gives $\theta = \frac{\pi}{2}$. The lines of force which touch the sphere on the equator are then tangents to the surface.

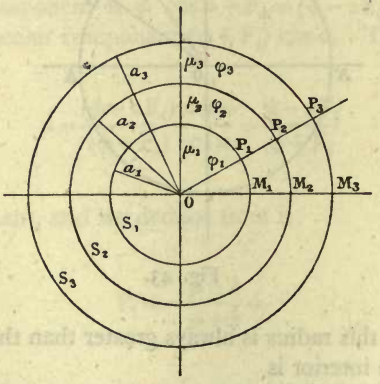

Fig. 44.

164. Concentric Spherical Layers in a Uniform Field.— It is easy to generalise the preceding problem, and to apply it to a series of concentric spherical layers. In a uniform field of strength ϕ let there be a system of concentric spheres S_1, S_2, S_3.... (Fig. 44) having the radii a_1, a_2, a_3..., and the specific inductive capacities μ_1, μ_2, μ_3...

Let us consider the inner sphere S_1. If the medium of specific inductive capacity μ_2, which surrounds it, were unlimited and formed a uniform field of strength ϕ_2, this sphere would be covered with a layer of displacement M_1, giving in the interior a constant force F_1 and a uniform field $\phi_1 = \phi_2 + F_1$; and for a point P_1 on the surface in the direction ω, we should have the equation

$$\mu_1\phi_1 \cos\omega = \mu_2(\phi_2 - 2F_1)\cos\omega,$$

or

$$\mu_1\phi_1 = \mu_2(\phi_2 - 2F_1). \tag{25}$$

But the uniform field of strength ϕ_2, situate on the outside of the sphere S_1, is that which would be produced for the interior of the sphere S_2 by an external uniform field of strength ϕ_3, and by the internal force F_2 due to the fictive layer distributed on the surface according to the same law, which would give

$$\phi_2 = \phi_3 + F_2.$$

For a point P_2 of this surface we have to consider not merely the action ϕ_3 of the external field, and that of the layer M_2, but also the action of the layer M_1 of the internal sphere S_1.

The law of the conservation of the flow of induction would give a relation between these quantities analogous to the equation (25) and which we may write directly in the following manner, suppressing the common factor $\cos\omega$:

$$\mu_2\left[\phi_2 - 2F_1\left(\frac{a_1}{a_2}\right)^3\right] = \mu_3\left[\phi_3 - 2F_2 - 2F_1\left(\frac{a_1}{a_2}\right)^3\right],$$

whence

$$\mu_2\phi_2 = \mu_3(\phi_3 - 2F_2) + 2(\mu_2 - \mu_3)F_1\left(\frac{a_1}{a_2}\right)^3. \tag{26}$$

The same reasoning applies to surface S_3; for a point P_3 of its surface we should then have to take into account the strength ϕ_1 of the external field, together with the actions of the three internal layers M_3, M_2 and M_1.

We shall thus have

$$\mu_3\left[\phi_3 - 2F_2\left(\frac{a_2}{a_3}\right)^3 - 2F_1\left(\frac{a_1}{a_3}\right)^3\right]\mu_4 = \left[\phi_4 - 2F_3 - 2F_2\left(\frac{a_2}{a_3}\right)^3 - 2F_1\left(\frac{a_1}{a_3}\right)^3\right]$$

Whence

$$(27) \qquad \mu_3\phi_3=\mu_4(\phi_4-2F_3)+2(\mu_3-\mu_4)\left[F_2\left(\frac{a_2}{a_3}\right)^3+F_1\left(\frac{a_1}{a_3}\right)^3\right].$$

The law of the terms is evident. Connecting these equations with the identities

$$(28) \qquad \begin{aligned} \phi_1&=\phi_2+F_1,\\ \phi_2&=\phi_3+F_2,\\ \phi_3&=\phi_4+F_3,\\ &\vdots\quad\vdots\quad\vdots \end{aligned}$$

we could determine the values of F_1, F_2, $F_3\ldots$ The problem is thus completely solved.

165. Let us suppose that there are two layers bounded by the surfaces S_1 and S_2, in a uniform external field of strength ϕ, where the dielectric is air. We may simply put

$$\phi_3=\phi,$$
$$\mu_3=1.$$

Equations (28) give then

$$\phi_2=\phi+F_2,$$
$$\phi_1=\phi+F_2+F_1.$$

Substituting in (25) and (26), and putting $\beta=\left(\frac{a_1}{a_2}\right)^3$, we have

$$(29) \qquad \begin{aligned} \mu_1(\phi+F_2+F_1)&=\mu_2(\phi+F_2-2F_1),\\ \mu_2(\phi+F_2)&=(\phi-2F_2)+2(\mu_2-1)\beta F_1, \end{aligned}$$

equations which would determine the forces F_1 and F_2 as functions of the data of the problem.

166. Suppose, further, that the internal nucleus is also air, which would amount to determining the state of a spherical layer; μ_1 must also equal 1. Let us denote by μ the specific inductive capacity of the medium previously denoted by μ_2, equations (29) would become

$$F_1(1+2\mu)=(\mu-1)(\phi+F_2),$$
$$F_2(\mu+2)\ =(\mu-1)(2\beta F_1-\phi).$$

From this is deduced

$$F_1[1+2\mu-2(\mu-1)\beta]=-3F_2,$$
$$F_1[(1+2\mu)(\mu+2)-2(\mu-1)^2\beta]=3(\mu-1)\phi,$$

and therefore

$$\frac{F_1}{3(\mu-1)} = \frac{-F_2}{(\mu-1)[1+2\mu-2(\mu-1)\beta]} = \frac{\phi}{(1+2\mu)(\mu+2)-2(\mu-1)^2\beta}.$$

The force in the interior at the surface S_1 is

$$F_i = \phi_1 = \phi + F_2 + F_1 = \phi \frac{9\mu}{(1+2\mu)(\mu+2)-2(\mu-1)^2\beta},$$

$$F_i = \frac{\phi}{1+\frac{2(\mu-1)^2}{9\mu}(1-\beta)}.$$

The force is constant inside S_1, but it is not constant between S_1 and S_2, nor outside S_2. The value of the force in the interior of S_1 is a fraction of the strength of the field, which would be equal to unity for $\mu = 1$, and to zero for $\mu = \infty$. With dielectrics whose coefficient μ does not differ much from 2, the fraction is always very near unity; if the layer is a conductor, the coefficient μ may be considered as infinite, and F_1 becomes zero.

We shall afterwards see the importance of this question in magnetism.

167. POISSON'S HYPOTHESIS ON THE CONSTITUTION OF DIELECTRICS.—Poisson's hypothesis, as revived by Faraday for electricity, consists, as we have already said (110), in assuming the dielectric to be formed of small conducting spheres disseminated in an insulating medium.

The results already obtained enable us to explain the method adopted by Poisson for calculating, at any rate approximately, the consequences of his hypothesis.

Consider a sphere of radius a_1, and of specific inductive capacity μ_1, situate in a field of strength ϕ_2, and of specific inductive capacity μ_2; from equations (25) and (28) the force on the interior of the electric layer is

$$F_1 = \phi_2 \frac{\mu_2 - \mu_1}{\mu_1 + 2\mu_2},$$

and the external potential of this layer on a point at a distance r is equal to

$$V_e = -F_1 \frac{a_1^3}{r^2} \cos\omega.$$

Let us suppose that a sphere of radius a contains a large number

of small spheres of radius a_1, and assume with Poisson that the electrification of each of them is not influenced by the electrification of the adjacent spheres, and only depends on the strength of the field. If n is the number of small spheres contained in the large one, the value of the potential at a distance, which is very great compared with a, is

$$V = nV_e = -F_1 \frac{na_1^3}{r^2} \cos\omega = -\phi_2 \frac{\mu_2 - \mu_1}{\mu_1 + 2\mu_2} na_1^3 \frac{\cos\omega}{r^2},$$

or putting $h = \frac{na_1^3}{a^3}$, that is to say calling h the ratio of the space, occupied by the small spheres, to the volume of the whole sphere,

$$V = -\phi_2 \frac{\mu_2 - \mu_1}{\mu_1 + 2\mu_2} ha^3 \frac{\cos\omega}{r^2}.$$

If the sphere were homogeneous, and of the specific inductive capacity μ, the potential V' at the same distance would be

$$V' = -\phi_2 \frac{\mu_2 - \mu}{\mu + 2\mu_2} a^3 \frac{\cos\omega}{r^2}.$$

The action of the two systems is identical if we have

$$\frac{\mu_2 - \mu}{\mu + 2\mu_2} = h \frac{\mu_2 - \mu_1}{\mu_1 + 2\mu_2},$$

which gives

$$\mu = \mu_2 \frac{\mu_1 + 2\mu_2 - 2h(\mu_2 - \mu_1)}{\mu_1 + 2\mu_2 + h(\mu_2 - \mu_1)};$$

this value of μ represents the *apparent* inductive capacity of the sphere made up as we have supposed.

If we assume that the small spheres are conductors, we must make $\mu_1 = \infty$; we have then

$$\mu = \mu_2 \frac{1 + 2h}{1 - h}.$$

If, finally, the external medium is of air, $\mu_2 = 1$, and we get

$$\mu = \frac{1+2h}{1-h}, \quad \text{or} \quad h = \frac{\mu - 1}{\mu + 2}.$$

For those dielectrics, whose specific inductive capacity is near 2, we should have

$$h = \frac{2-1}{2+2} = \frac{1}{4}.$$

This result may give some idea of the degree of exactitude to which Poisson's reasoning tends.

In a conducting sphere, the interior force due to the induced layers is equal to the action of the external field. The external action of a polarized sphere is very small compared with the internal action, for the ratio of the forces (158) is at most equal to $2\left(\frac{a}{r}\right)^3$ and tends towards zero when the spheres are infinitely small. But if the volume occupied by the conducting spheres is a quarter of the total volume, the action which each of them exerts upon the adjacent ones can no longer be neglected in comparison with the internal force, and the field is thus modified. The maximum ratio of the sum of the volumes of the spheres which touch, to the total space is equal to $\frac{\pi}{3\sqrt{2}}$ or sensibly $\frac{1}{\sqrt{2}}$. If this ratio is reduced to $\frac{1}{4}$, the distance of the centres of two adjacent spheres is about equal to the diameter multiplied by $\sqrt[3]{\frac{4}{\sqrt{2}}}$. The action exerted by the electrical layer of one of the spheres, at the centre of the nearest one, might thus attain a fraction of the internal force equal to

$$2\left(\frac{a}{2a\sqrt[3]{\frac{4}{\sqrt{2}}}}\right)^3 = \frac{\sqrt{2}}{16} = \frac{1}{11}.$$

It is true that if the reciprocal action of the sphere tends to increase the electrification parallel to the force of the field, it tends to diminish it in a perpendicular direction, so that we are not far from the truth in assuming, with Poisson, that this reciprocal influence may be neglected.

168. Two Unequal Masses of Opposite Signs.—Let $+m$ and $-m'$ be two masses of opposite signs situate at A and A′ (Fig. 45) at a distance of $2a$, m being greater than m' in absolute value; let us put

$$\frac{m}{m'} = k^2.$$

The equation of an equipotential surface is

$$V = \frac{m}{r} - \frac{m'}{r'} = m'\left(\frac{k^2}{r} - \frac{1}{r'}\right) = \frac{m'}{r}\left(k^2 - \frac{r}{r'}\right) = \frac{m}{r'}\left(\frac{r'}{r} - \frac{k^2}{1}\right).$$

One of these surfaces corresponds to potential zero, and its equation is

$$\frac{m}{r} - \frac{m'}{r'} = 0,$$

whence

$$\frac{r}{r'} = \frac{m}{m'} = k^2$$

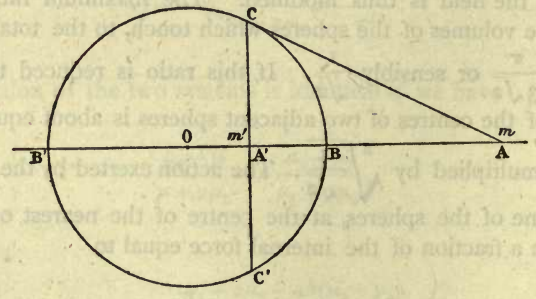

Fig. 45.

It is a sphere to which the point A′ is internal; the two points A and A′ are conjugate in reference to this sphere.

To determine the radius R, and the centre O of the sphere, we shall use the ratios

$$k^2 = \frac{BA}{BA'} = \frac{B'A}{B'A'} = \frac{R}{OA'} = \frac{OA}{R}$$

Remarking that $OA - OA' = 2a$, we easily deduce from this

$$R = 2a\frac{k^2}{k^4 - 1},$$

$$(30) \qquad OA = 2a\frac{k^4}{k^4 - 1},$$

$$OA' = 2a\frac{1}{k^4 - 1}$$

The potentials are negative inside the sphere, and positive outside.

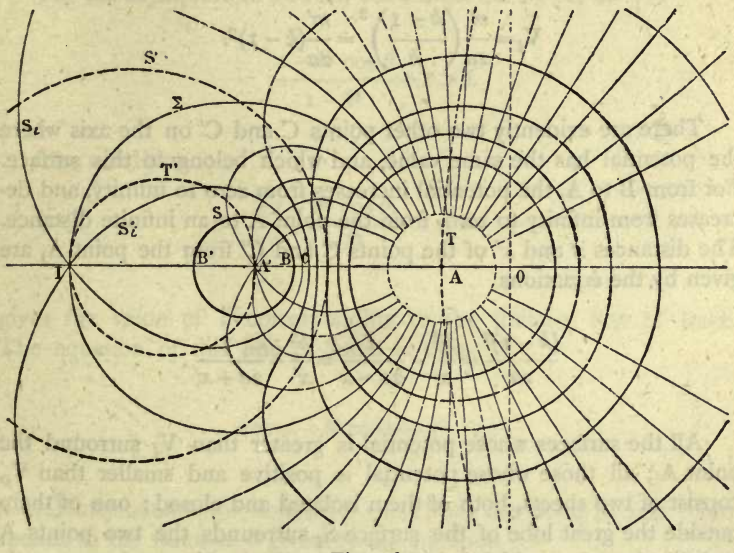

Fig. 46.

All the equipotential surfaces are closed surfaces with one sheet or with two distinct sheets, except a single one which has two adjacent sheets S_i and S'_i, and which passes through the point I (Fig. 46) where the force is zero. The position of this point is given by the equation

$$\frac{m}{\overline{IA}^2} = \frac{m'}{\overline{IA'}^2},$$

or

$$\left(\frac{IA}{IA'}\right)^2 = \frac{m}{m'} = k^2;$$

we have thus

$$\frac{IA}{IA'} = k,$$

and therefore

$$IA = 2a\frac{k}{k-1},$$

$$IA' = 2a\frac{1}{k-1}.$$

The value of the potential at I, and on the whole surface with two sheets, is

$$V_i = \frac{m}{2a}\left(\frac{k-1}{k}\right)^2 = \frac{m'}{2a}(k-1)^2.$$

There are evidently two other points C and C′ on the axis where the potential has the same value, and which belong to this surface. For from B to A, the potential increases from zero to infinity, and decreases from infinity to zero, from the point A to an infinite distance. The distances x and x' of the points C and C′ from the point A, are given by the equations

$$\frac{(k-1)^2}{2a} = \frac{k^2}{x} - \frac{1}{2a-x} = \frac{k^2}{x'} - \frac{1}{2a+x'}.$$

All the surfaces whose potential is greater than V_i surround the point A; all those whose potential is positive and smaller than V_i, consist of two sheets, both of them isolated and closed; one of them outside the great lobe of the surface S_i surrounds the two points A and A′; the other, which is inside the small lobe S'_i, merely surrounds the point A′.

169. The general equation of the lines of force

$$m\cos\omega + m'\cos\omega' = m' + m\cos\theta$$

becomes here

$$m\cos\omega - m'\cos\omega' = -m' + m\cos\theta,$$

or

$$\cos\omega' - k^2\cos\omega = 1 - k^2\cos\theta. \qquad (31)$$

The asymptote which corresponds to $\omega = \omega'$, is defined by the equation

$$(1 - k^2)\cos\alpha = 1 - k^2\cos\theta;$$

it passes through the centre of gravity O of the two masses. This point is given by the ratio

$$\frac{OA}{OA'} = \frac{m'}{m} = \frac{1}{k^2}.$$

For the asymptote to be real, $\cos\alpha$ must be > -1, or

$$\frac{1 - k^2\cos\theta}{1 - k^2} + 1 > 0.$$

The condition

$$\frac{1 - k^2\cos\theta}{1 - k^2} + 1 = 0$$

gives the value of θ corresponding to the limiting line of force. The equation of this line of force is then

$$\cos\omega' - k^2\cos\omega = k^2 - 1;$$

it evidently bounds the flow $4\pi(m - m')$ from the point A, and corresponds to the value of θ given by the equation

$$1 - \cos\theta = \frac{N}{2\pi m} = \frac{2(m - m')}{m} = 2\left(1 - \frac{1}{k^2}\right),$$

or

$$\cos\theta = \frac{2}{k^2} - 1.$$

This line of force Σ passes moreover through the point of equilibrium I, as can be easily shown.

We can determine the direction of the tangent as above (145), which gives

$$\frac{\sin\beta}{\sin\beta'}=\frac{f'}{f}=\frac{m'r^2}{mr'^2}=\frac{1}{k^2}\cdot\frac{r^2}{r'^2}. \tag{32}$$

170. When the tangent is horizontal, we have

$$\beta+\omega=\pi,$$
$$\beta'=\omega',$$

or

$$\sin\beta=\sin\omega,$$
$$\sin\beta'=\sin\omega'.$$

Equation (32) becomes then

$$\frac{\sin\omega}{\sin\omega'}=\frac{1}{k^2}\cdot\frac{r^2}{r'^2}.$$

or replacing the sines by the opposite sides,

$$\frac{r'}{r}=\frac{1}{k^2}\cdot\frac{r^2}{r'^2},$$

which equation may be thus written

$$\frac{r}{r'}=k^{\frac{2}{3}}.$$

The locus of the points where the tangent is horizontal, is therefore a sphere comprising the point A'. The centre and the radius of this sphere may be calculated by formulas (30) in which k is replaced by $k^{\frac{1}{3}}$.

171. When the tangent is vertical, we have

$$\omega+\beta=\frac{\pi}{2},$$

$$\beta'-\omega'=\frac{\pi}{2},$$

or

$$\sin\beta=\cos\omega.$$
$$\sin\beta'=\cos\omega'.$$

In this case equation (32) becomes

$$\frac{r^2}{\cos\omega} = k^2 \frac{r'^2}{\cos\omega'};$$

it represents a curve formed of two branches, one proceeding from the point A, the other from the point A′ (Fig. 46).

The branch T proceeding from A is at first vertical at this point. For points at a considerable distance, the angles ω and ω' tend towards equality, and we have

$$\frac{\cos\omega}{r^2} = \frac{\cos\omega'}{k^2 r'^2} = \frac{\cos\omega - \cos\omega'}{r^2 - k^2 r'^2} = \frac{\sin\omega \,.\, \delta r}{r(r^2 - k^2 r^2)} = \frac{2a\sin^2\omega}{r^3(1-k^2)},$$

from which

$$\frac{\sin^2\omega}{\cos\omega} = \frac{r(1-k^2)}{2a}.$$

The second member increases to infinity with r. The angle ω tends then towards $\frac{\pi}{2}$, and the curve has a vertical asymptote which evidently passes through the point O.

The second branch is a closed curve T′; it passes through the point A′, and through the point I.

172. Electrification of a Sphere under the Influence of a Point.—We know from the theorems already proved (61), that we can replace the mass $-m'$ by an equal layer in equilibrium on any one of the equipotential surfaces which surround the point A′, comprising the sheet S'_i of the surface with two sheets. In like manner we may replace the mass m by an equal layer on one of the equipotential surfaces which surround A, including the surface S_i. The two masses m and $-m'$ may, lastly, always be replaced, for external points, by one mass $m - m'$ on one of the surfaces which surround the two points, including again the surface S_i.

If, in particular, we consider the sphere S of potential zero, which surrounds the point A′, we can replace $-m'$ by an equal mass in equilibrium on the sphere. Nothing will be changed for external points; but for points in the interior the potential will be constant and equal to the value which it has on this surface—that is to say, zero. For points inside the surface S, the mass m may be replaced by a mass $+m'$ in equilibrium on this surface, and thus the potential will everywhere be zero on the outside.

The first case corresponds to the electrification of an uninsulated sphere under the influence of an external mass; the second gives the influence of an electrified mass on an uninsulated spherical surface which surrounds it.

The density σ of the layer at each point should satisfy the ratio

$$F = -4\pi\sigma.$$

At the point P, on the surface (Fig. 47) the force is directed along PO; it is the resultant of the forces f and f', one acting from A and

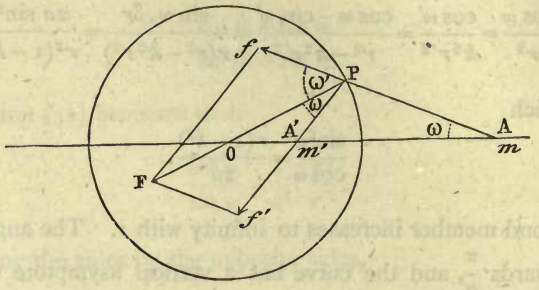

Fig. 47.

the other directed towards A'. The triangle formed by the three forces F, f, and f' is similar to the triangle APA'; we have then

$$\frac{F}{AA'} = \frac{f}{r'} = \frac{f'}{r},$$

and, consequently,

$$F = 2a\frac{f}{r'} = 2a\frac{m}{r^2 r'} = 2ak^2\frac{m}{r^3} = \frac{2a}{k^2}\cdot\frac{m'}{r'^3}.$$

From this we have

$$\sigma = -\frac{2ak^2}{4\pi}\cdot\frac{m}{r^3} = -\frac{2a}{4\pi k^2}\cdot\frac{m'}{r'^3}.$$

The force and density at a point on the surface S are therefore inversely as the cube of the distance, either from the point A, or from the point A'.

This density is positive if the mass m be replaced by a layer distributed on the spherical surface S; this is the case of the inductive action of a mass $-m'$, placed inside an uninsulated spherical surface.

The density is negative if the action of this layer be substituted for that of the mass $-m'$, which corresponds to the problem of an uninsulated sphere S under the influence of the mass m at A.

In these two cases, the sphere is given as well as the position and magnitude of one of the masses.

Knowing the mass m, the radius of the sphere R, and the distance $AO = d$, we get directly

$$k^2 = \frac{d}{R}, \qquad m' = -m\frac{R}{d}, \qquad 2ad = d^2 - R^2,$$

and, consequently,

$$\sigma = -\frac{d^2 - R^2}{4\pi R} \cdot \frac{m}{r^3}.$$

173. If, after having insulated the sphere, we superpose on the layer $-m$ any uniform layer M, equilibrium still holds, and the potential of the sphere, which was zero, becomes

$$V_1 = \frac{M}{R}.$$

If we make $M = m'$, the total charge of the sphere is zero, and its potential is

$$V_0 = \frac{m'}{R}.$$

This is the case of an insulated sphere, originally in the neutral state, electrified under the influence of an external point. As the mass is null, the potential at the centre only depends on the external mass. We must then have

$$V_0 = \frac{m'}{R} = \frac{m}{d}.$$

The density of this new layer being

$$\sigma_0 = \frac{m'}{4\pi R^2} = \frac{m}{4\pi R d},$$

the resultant density is

$$\sigma_0+\sigma=\frac{m}{4\pi Rd}-\frac{d^2-R^2}{4\pi R}\cdot\frac{m}{r^3}=\frac{m}{4\pi R}\left[\frac{1}{d}-\frac{d^2-R^2}{r^3}\right].$$

The density will be zero for all points of the small circle perpendicular to the axis defined by the equation

$$r^3=d(d^2-R^2).$$

The plane of this small circle cuts the axis OA on the left of the point A', since we have $r>\sqrt{d^2-R^2}$. It is the *neutral line* which separates the positive from the negative zone. It is a line of equilibrium—the force and the density there are null. It is the intersection, by the sphere, of the equipotential surface $V_0=\frac{m}{d}$; we know, moreover, that the two surfaces intersect at a right angle.

The density will be null on the small circle, formed by the contact of the tangent cone to the sphere, and having its apex at the point A, the plane of which circle passes through the point A', provided that

$$\frac{M}{4\pi R^2}=\frac{d^2-R^2}{4\pi R}\cdot\frac{m}{(d^2-R^2)^{\frac{3}{2}}}=\frac{m}{4\pi R\sqrt{d^2-R^2}}.$$

and therefore

$$\frac{M}{m}=\frac{R}{\sqrt{d^2-R^2}}.$$

174. The action of the insulated sphere S, electrified by induction from the mass m, on all external points, may be replaced by that of a mass $-m'=-\frac{m}{k}$, placed at A'. In like manner the uninsulated surface S' acted on by induction from $-m'$, is equivalent for all internal points to the mass $m=km'$ placed at A.

175. IMAGE OF ANY GIVEN SYSTEM.—The principle of images in reference to a sphere may be extended to any system whatever—for instance, to an electrified layer. For each element of the systems develops by induction, on the sphere, a layer whose action on external points is identical with that of the corresponding image. As each of

these layers is in equilibrium, their superposition will be a state of equilibrium, and the resultant action will be equal to the resultant action of all the images. The totality of these images will form a system, which is the image, in reference to the sphere of the given system. If the given system is a surface Σ, the image will be a surface conjugate to the first.

176. Reciprocal Action of Two Spheres.—The principle of images combined with Murphy's method (86), enables us to solve completely the very important problem of the reciprocal action of two spheres. Let S_a and S_b be the two spheres (Fig. 48), R and R′ the radii. The method consists, as we know, in determining a series of successive layers in the following manner. On the conductor S_a a layer is placed capable of giving the potential 1; this is a uniform layer of mass R. This layer acts outwards as if it were concentrated at A. It is fixed and the induced layer on the surface S_b of the

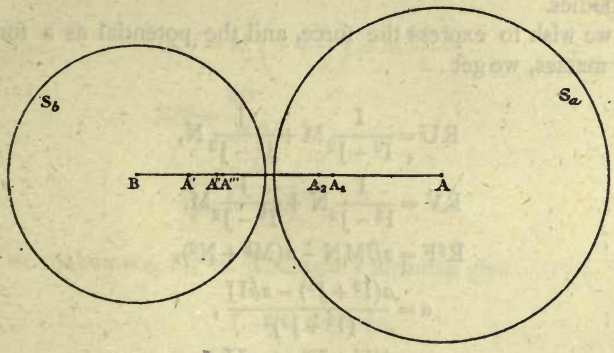

Fig. 48.

second uninsulated sphere is determined, which amounts to determining the image A′ in reference to S_b of a mass + R at A. The equivalent layer is next fixed at A′, and its inductive action on the uninsulated sphere S_a is determined—that is to say, the new image A_1 of A′, and so on. The same operation will be repeated beginning with the sphere S_b, and all the masses thus determined are multiplied by suitable coefficients. As each of the masses and the densities can be exactly calculated, the problem of distribution is completely solved.

The force exerted between the two spheres is the resultant of the actions exerted by each of the masses comprised within one of the spheres on all the masses contained in the second.

The calculation does not present any theoretical difficulties, but it is very tedious. Sir W. Thomson performed it in the case of two spheres of the same radius when the distance of the centres varies between 2R and 4R, that is to say, when the distance of the surfaces is comprised between 0 and the diameter of one of the spheres.

In the present case, if R is the common radius of the spheres A and B, U and V the potentials, cR the distance of the centres, M and N the respective charges, then if I, J, a and b are coefficients which depend on c, we have

$$\begin{aligned} M &= R(IU - JV), \\ N &= R(IV - JU), \\ F &= 2bUV - a(U^2 + V^2), \end{aligned}$$

expressions analogous to those furnished by Murphy's method for any given bodies.

If we wish to express the force, and the potential as a function of the masses, we get

$$\begin{aligned} RU &= \frac{I}{I^2 - J^2} M + \frac{J}{I^2 - J^2} N, \\ RV &= \frac{I}{I^2 - J^2} N + \frac{J}{I^2 - J^2} M. \\ R^2F &= 2\beta MN - \alpha(M^2 + N^2), \\ \alpha &= \frac{a(I^2 + J^2) - 2bIJ}{(I^2 - J^2)^2}, \\ \beta &= \frac{b(I^2 + J^2) - 2aIJ}{(I^2 - J^2)^2}. \end{aligned}$$

If the charges M and N are equal, we get

$$\begin{aligned} M &= RV(I - J), \\ F &= 2(b - a)V^2, \\ RV &= \frac{M}{I - J}. \\ R^2F &= 2(\beta - \alpha)M^2. \end{aligned}$$

177. As these formulæ have only hitherto been calculated for $c = 4$, it is useful to see how they may be replaced for greater distances.

Suppose that the action of the two spheres is the same as if the masses were respectively concentrated at the centres, and that the potential of each of them is equal to that obtained for the centre, by replacing the adjacent sphere by an equal mass situate at the centre. We shall have thus

$$U = \frac{M}{R} + \frac{N}{cR}.$$

$$V = \frac{N}{R} + \frac{M}{cR},$$

$$F = \frac{MN}{c^2R^2},$$

$$F = \frac{c^2+1}{(c^2-1)^2} UV - \frac{c}{(c^2-1)^2}(U^2+V^2);$$

and for equal charges

$$RV = M\left(1+\frac{1}{c}\right) = M\frac{c+1}{c},$$

$$R^2F = \frac{M^2}{c^2},$$

$$F = \frac{V^2}{(c+1)^2}.$$

If we make $c=4$, Sir W. Thomson's formulæ give

$$I - J = 0{\cdot}80258,$$
$$2(\beta-\alpha) = 0{\cdot}05846,$$
$$2(b-a) = 0{\cdot}03766.$$

The corresponding values in the approximate formula are

$$\frac{c}{c+1} = \frac{4}{5} = 0{\cdot}80,$$

$$\frac{1}{c^2} = \frac{1}{16} = 0{\cdot}0625,$$

$$\frac{1}{(c+1)^2} = \frac{1}{25} = 0{\cdot}04.$$

Thus, when the distance of the surfaces is equal to the diameter

of one of the spheres, and the charges are equal, the approximate formulæ give the potentials as a function of the masses, the value of the force as a function of the masses, and the force as a function of potentials with a degree of approximation which is $\frac{1}{300}$, $\frac{1}{16}$ and $\frac{1}{17}$ for the various forces respectively. As the relative errors amount to $\frac{1}{12}$, when $c=3\cdot8$, it will be seen that they will diminish very rapidly for greater distances.

We may therefore, as a particular case, regard Coulomb's method for determining the action of two electrified masses as perfectly exact; and Coulomb's method will give with the approximate formulæ very exact results for the measurement of potentials, if we take care that the distance of the surfaces of the spheres sensibly exceeds the diameter of one of them. In determining potentials by means of the balance, we ought always to take special precautions for eliminating or for calculating the influence of the sides of the cage.

178. Motion of Small Bodies in the Electrical Field.—We have seen that (162) a dielectric sphere of radius a, placed in a uniform field where the force is ϕ, becomes electrified so that the internal action of the equivalent layer is

$$F_i = -\frac{\mu-1}{\mu+2}\phi = -h\phi.$$

This coefficient h is equal to $\frac{1}{4}$ for the dielectrics whose specific inductive capacity is 2, and it is equal to unity for conductors.

From this we deduce (159)

$$\sigma_0 = \frac{3h}{4\pi}\phi$$

$$\varpi = u\sigma_0 = u\frac{3h}{4\pi}\phi = uK\phi.$$

The coefficient K is equal to $\frac{3}{4\pi}$ for conductors, and has a smaller value for dielectrics, since the factor h is always less than unity.

Consider now a sphere so small that, placed at any point of a variable field, it becomes electrified as it would be in a uniform field where the force was the same; the electrical moment of this sphere will be

$$\varpi = u\sigma_0 = u\mathrm{K}\phi,$$

and the axis of electrification being parallel to the force ϕ, will be perpendicular to the equipotential surfaces which include it.

Suppose that we fix on the surface the two electrical layers, which are equivalent to the system of two solid spheres, or of two equal and opposite masses $\pm m$ at an infinitely small distance δ, such that $\varpi = m\delta$. If V_1 and V_2 are the potentials at the points occupied by the masses $-m$ and $+m$, the energy of this sphere in the field is then

$$W = \int mV = -m(V_1 - V_2) = -m\delta\,\frac{V_1 - V_2}{\delta} = -\varpi\phi.$$

At a point P′ where the force is equal to $\phi + d\phi$, the energy of the sphere will be

$$W' = -\varpi(\phi + d\phi).$$

The work necessary for bringing the sphere from the point P to the point P′ is then

$$W' - W = -\varpi d\phi.$$

In reality, if the layers are not fixed, the electrification changes with the displacement of the sphere, and the work in question is between $-\varpi d\phi$ and $-(\varpi + d\varpi)d\phi$; this work only differs therefore from the value found by an infinitely small expression of the second degree.

The energy dW expended in effecting the displacement is then

$$dW = -\varpi d\phi = -u\mathrm{K}\phi d\phi,$$

(33)
$$dW = -\frac{u\mathrm{K}}{2}d(\phi^2).$$

Thus, when an infinitely small sphere passes from a point in the

field where the force is ϕ_0, to another where the force is ϕ, the increase of energy is

$$W - W_0 = -\frac{uK}{2}(\phi^2 - \phi_0^2).$$

If the sphere is brought from an infinite distance, we shall have

$$W = -\frac{uK}{2}\phi^2.$$

The sphere left to itself tends of course to expend energy, and therefore, by equation (33), to move in a direction in which the value of ϕ^2 increases most rapidly. It tends then to move to points in which the force is a maximum in absolute value.

179. Let n be the direction in which ϕ^2 varies most rapidly; the expression for the force acting on the sphere is

$$F = -\frac{dW}{dn} = \frac{uK}{2} \cdot \frac{d\phi^2}{dn}. \tag{34}$$

and its components parallel to the co-ordinates are

$$X = \frac{uK}{2} \cdot \frac{\partial\phi^2}{\partial x},$$

$$Y = \frac{uK}{2} \cdot \frac{\partial\phi^2}{\partial y}. \tag{35}$$

$$Z = \frac{uK}{2} \cdot \frac{\partial\phi^2}{\partial z}.$$

180. *In a variable electrical field, the force cannot be a maximum at any point situate outside the acting masses.*

This theorem follows directly from the preceding demonstration.

We have seen in fact from Earnshaw's theorem (63), that an electrified body cannot be in stable equilibrium in a variable field. Since an infinitely small sphere can only be in stable equilibrium at points where the value of ϕ^2 is an absolute maximum, that is to say, where the value of ϕ is a maximum in absolute value, it follows that this circumstance cannot present itself for any point outside the acting masses.

181. The same reasoning applies to the motion of a very small body of any given form, if we neglect the effects of rotation—that is to say, if we assume that the body always retains the same direction in reference to the lines of force. This body, in fact, becomes electrified proportionally to the force of that point of the field in which it is situated, and the variation of energy is proportional to the variation of the square of the force.

Independently of this progressive motion, a body which is not spherical will turn on itself in every point, in such a manner that for stable equilibrium about its centre of gravity the electrical energy is a minimum, and the electrification is a maximum.

Such, according to Sir W. Thomson, is the true meaning of the attraction of light bodies of small dimensions in an electrical field, so long at any rate as they have not been electrified by direct contact. These bodies, whether conductors or not, move towards points where the force is a maximum in absolute value, and they finish by coming in contact with the electrified surfaces. If they were movable in a medium in which an extraneous resistance would keep the velocity very small, they would move towards the electrified body, not along a line of force, but along a line of maximum variation of the force; in certain cases, in which the body is impeded, this motion may even be perpendicular to the force.

The body is in equilibrium for points in which we have

$$d\phi^2 = 0, \qquad \text{or} \qquad \phi d\phi = 0.$$

This condition may be realised in two ways

$$\phi = 0, \qquad \text{or} \qquad d\phi = 0.$$

Hence there is equilibrium when the force is null, maximum, minimum, or stationary. There is equilibrium particularly in a uniform field, which was *à priori* evident. The equilibrium is thus neutral; it is stable at the maxima of force, unstable at the minima, and at the points at which the force is null.

182. Direction of a Dielectric Needle in a Variable Field.—Let us suppose that on a sphere charged, as we have always supposed, by layers of displacement, we cause a force F to act which is constant in magnitude and direction, and which makes the angle θ with the axis of electrification; the moment of the couple produced by this force will be

$$Fm.\delta \cos\theta = F\varpi \cos\theta.$$

The electrical moment ϖ is therefore the moment of the couple which this sphere would experience in a field equal to unity, the force of which is perpendicular to the axis of electrification.

If we adopt the conception of Poisson and of Faraday regarding the constitution of dielectrics, and consider them as formed of conducting spheres disseminated in an insulating medium; if we admit, further, that the electrification of each of them is not modified by the adjacent ones, the electrical moment of a body of any given form in a uniform field is

$$\varpi = nu\frac{3}{4\pi}\phi = \frac{nu}{\mathrm{U}}\mathrm{U}\frac{3}{4\pi}\phi = \mathrm{U}\frac{3h}{4\pi}\phi = \mathrm{UK}\phi,$$

where U is the volume of the dielectric, n the number of spheres which it contains, and u the volume of each one of them.

The expression of this moment is therefore the same as for a homogeneous sphere.

On this hypothesis, a body of any given form in a uniform field would also be in equilibrium in reference to its centre of gravity, whatever was its direction. For all the volume-elements become electrified parallel to the force of the field, and the couple of rotation, being null on each of them, is null upon the whole.

In a variable field, on the contrary, a very small body, fixed by its centre of gravity, tends to take a certain direction. As each volume-element du is only acted on by the force of the field, it tends to move towards points where the force increases, and the components of the force which it undergoes are

$$\mathrm{X} = \frac{\mathrm{K}du}{2}\frac{\partial\phi^2}{\partial x},$$

$$\mathrm{Y} = \frac{\mathrm{K}du}{2}\frac{\partial\phi^2}{\partial y},$$

$$\mathrm{Z} = \frac{\mathrm{K}du}{2}\frac{\partial\phi^2}{\partial z}.$$

183. Consider a short and infinitely thin needle, and let ϕ_0 be the force of the field at the centre of gravity O of the needle. Let us take the direction of the force for the x-axis; suppose that the needle can turn about the z-axis, and that it makes the angle θ with ϕ_0.

For the volume element du at a distance a from the centre, and

whose co-ordinates are x and y, the component ϕ of the force, parallel to the plane, will have an expression of the form

$$\phi^2 = \phi_0^2 + Ax + By.$$

The components X and Y of the force will be

$$X = \frac{K du}{2} A,$$

$$Y = \frac{K du}{2} B.$$

The component tangential to the circle which the volume-element du describes is

$$T = X \sin\theta - Y \cos\theta = \frac{K du}{2}(A \sin\theta - B \cos\theta).$$

The position of equilibrium corresponds to the condition

$$A \sin\theta - B \cos\theta = 0, \qquad \text{or} \qquad \tan\theta = \frac{B}{A},$$

that is to say, to the direction along which the variation of the force is a maximum.

When the element is turned through an angle $d\theta$ from its position of equilibrium, the tangential component is

$$dT = \frac{K du}{2}(A \cos\theta + B \sin\theta) d\theta,$$

the angle θ being determined by the condition of equilibrium; from this is deduced

$$A \cos\theta + B \sin\theta = \sqrt{A^2 + B^2}.$$

We have then

$$dT = \frac{K du}{2} \sqrt{A^2 + B^2}\, d\theta.$$

For the entire needle, the moment of the resultant in reference to the axis will be

$$\int ad\mathrm{T} = \frac{\mathrm{K}}{2}\int adu\,\sqrt{\mathrm{A}^2+\mathrm{B}^2}\,d\theta.$$

If ρ is the density of the substance in question, the time t, of infinitely small oscillations, is given by the equation

$$t^2 = \pi^2 \frac{\rho\int a^2 du}{\frac{\mathrm{K}}{2}\int adu\,\sqrt{\mathrm{A}^2+\mathrm{B}^2}}.$$

If the needle is rectilinear, and of the length $2l$, we have

$$\frac{\int a^2 du}{\int adu} = \frac{2}{3}l,$$

which gives finally

$$t^2 = \frac{4}{3}\pi^2 \frac{\rho l}{\mathrm{K}\sqrt{\mathrm{A}^2+\mathrm{B}^2}}.$$

Thus, in a variable field, a dielectric needle able to turn about its centre of gravity is directed, not along the line of force, but along the line of greatest variation of the force; the square of the time of oscillations is not simply in the inverse ratio of the force of the field, as it depends on the coefficients of the variation, and, other things being equal, is proportional to the length of the needle.

184. The phenomenon is particularly useful to consider when the field is symmetrical in reference to the point at which is the needle. Suppose, for instance, the needle to be placed at the middle point O of the distance AA′ (Fig. 34) between two equal masses of opposite signs, and that it can oscillate about a right line perpendicular to AA′. In the plane of oscillation, the force at the point O is a minimum for the direction A′A or Ox, and a maximum for the perpendicular direction Oy. For a point adjacent to O we may therefore write

$$\phi^2 = \phi_0^2 + \mathrm{A}x^2 - \mathrm{B}y^2.$$

The components of the force which are exerted on the element du are

$$X = K du A x = K du A a \cos\theta,$$
$$Y = -K du B y = -K du B a \sin\theta,$$

and the tangential component

$$T = aK du(A + B) \sin\theta \cos\theta.$$

The condition of equilibrium is thus

$$\sin\theta \cos\theta = 0,$$

which gives two directions, $\theta = 0$ and $\theta = \frac{\pi}{2}$, the first corresponding to stable, and the second to unstable, equilibrium.

For a very small deviation $d\theta$ from the position of stable equilibrium, the tangential component is

$$dT = aK du(A + B) d\theta.$$

If the volume-element is isolated, the duration of the oscillation is given by the formula

$$t^2 = \pi^2 \frac{a^2 \rho du}{aK du(A + B)a} = \pi^2 \frac{\rho}{K} \frac{1}{A + B};$$

it is thus seen to be independent of a, and to depend only on the density of the substance, on its electrical susceptibility, and on the law of the variation of the field.

If we collect on a straight line a series of similar particles, and if we suppose that they exert no influence on each other, each of them would act as if it were alone, and the oscillation of the whole needle would take place in the same time as each of its parts; the duration of the oscillation of a needle for the state of the field in question is therefore independent of its length.

185. ACTION OF A FIELD ON A CONDUCTING NEEDLE.—A conductor behaves in a totally different manner. Consider, for instance, an infinitely small needle, so that the electrification may be supposed identical with that which would be produced in a uniform field.

The needle is in equilibrium in a direction perpendicular to the line of force, but it is evident that then the electrification is a minimum and that equilibrium is unstable.

When the needle is oblique to the force of the field, the two positive and negative electrical layers are symmetrical in reference to the centre, and the action of the field evidently produces a couple, which tends to move the needle in the direction of the force.

The law of distribution is independent of the force of the field, and each of the layers is proportional to this force. Hence the moment of the couple which acts upon the needle, for a given deviation from its position of equilibrium, is proportional to the square of the force ϕ. Lastly, the duration of the oscillation is inversely as the force, and we may write

$$t = \frac{A}{\phi},$$

the constant A only depending on the ratio between the longitudinal and transverse dimensions of the needle.

CHAPTER VIII.

SOURCES OF ELECTRICITY.

186. We have already mentioned Volta's most important discovery that two conductors, and, more generally, any two bodies placed in contact, assume different electrical conditions on each side of the surfaces in contact.

In the case of two different conductors in contact and in equilibrium, the potential is constant on either of them, but experiences a sudden change, on passing from one surface to the other.

We need not mention here all the experiments by which this law, of such fundamental importance, has been established; we shall merely adduce the following experiment, which will serve to define the conditions of the phenomenon.

If two plates, one of zinc and the other of copper, in the neutral state and at the same temperature, while held by insulating handles, are placed in contact with their parallel faces, and are then separated from each other, each of them will be found to be electrified, the zinc positively and the copper negatively. The electrical charge on each of the plates is proportional, other things being equal, to the extent of the surfaces in contact.

The phenomenon is as if the plates were the coatings of a condenser, between which there existed a given difference of potential; the corresponding electrical layers being on each side of the surface of separation in the two metals respectively, and at a very small distance apart, so that the capacity of the system is simply proportional to the extent of the surfaces facing each other.

187. ELECTROMOTIVE FORCE OF CONTACT.—If V_1 and V_2 are the potentials of the zinc and of the copper, δV their difference $V_1 - V_2$, or the *electromotive force of contact*, and C the capacity of the condenser formed when they are in contact, the charge of the plates will be

$$m = C(V_1 - V_2) = C\delta V.$$

A charge of electricity M may be imparted to the system of the two plates, which raises the zinc to a potential V, for instance. In this case the same difference of potential is still maintained at the surface of contact; when the plates are separated the charge on each is made up of the charge $\pm m$ due to the difference of potential of contact, together with part of the common charge M, which has been divided between them according to the ordinary laws of distribution.

The experiment presents great difficulties when it is attempted to deduce the difference of potential from the magnitude of the charges; for the real distance of the plates depends on the degree of polish of the surfaces; the plates are often in contact in only a small number of points, and the capacity of the system may vary within considerable limits.

More regular results are obtained by keeping the plates parallel to each other at a certain distance e and connecting them by a wire of copper or of zinc. The difference of potential δV produced at the point of contact, is maintained over the whole extent of the two plates; the distance of the corresponding layers is sensibly equal to the thickness of the dielectric which separates them, and the capacity of the system is inversely as this thickness. If, after having suppressed the external contact, the plates are moved away from each other, then, if S is the extent of surface,

$$m = \frac{S}{4\pi e}\delta V.$$

This experiment will enable us to determine the absolute value of δV. In all cases, if a constant distance e is maintained between the plates, and if different metals are employed, the ratios of the electromotive forces of contact may be determined.

As a matter of fact, the experiments are very delicate, owing to alterations in the results by very slight modifications in the state of the surfaces. The very nature of the gas which constitutes the dielectric seems to have a slight influence; it may be that the layer of gas, adhering to the metal, changes its physical properties, or that some particular chemical compound is formed there.

188. VOLTA'S LAWS.—LAW OF CONTACT.—However this may be, Volta's ideas have been confirmed, by whatever progress has been made in electricity, and the following law may be enunciated as the *law of contact*:—

If two bodies are in contact at the same temperature, a finite

difference of potential is set up, which depends on their nature, and which is altogether independent of their dimensions, of their shape, of the extent of surface in contact, and of the absolute value of the potential on each of them.

Volta characterized this property by saying that there is a *tension of contact* between the two surfaces, but the manner in which he conceived the phenomenon is quite in agreement with the idea of a difference of potential.

We shall represent this characteristic difference of potential, or electromotive force of two metals A and B, by the symbol A|B, the first letter denoting the metal whose potential is highest. We have then

$$\delta V = V_a - V_b = A|B.$$

We may at once add that this difference is a function of the temperature, and that the contact of two bodies of the same kind, but at different temperatures, also gives rise to an electromotive force.

In all the questions relative to the electrical equilibrium of conductors, we have hitherto neglected the electromotive forces due to the contact of heterogeneous conductors. All the calculations presuppose that the conductors are identical and at the same temperature, and the results should be modified by allowing for this new circumstance, unless in the case of very high potentials, where the effects of contact may be neglected.

189. Law of Successive Contacts.—After having confirmed the fundamental fact of the electromotive force of contact, Volta compared with each other the results furnished by different metals, and established experimentally a second law, which may be thus enunciated:—

When several metals at the same temperature are soldered to each other so as to form a continuous chain, the difference of potentials of the extreme metals is the same as if these two metals are in direct contact.

Let A, B, C..... L, M be the metals constituting the chain; this law, with the symbols adopted above, is represented by the following formula:—

$$A|B + B|C..... + L|M = A|M.$$

We have, further,

$$A|M = -M|A.$$

By bringing all the terms within the first member, the equation becomes

$$A|B+B|C+\ldots\ldots+L|M+M|A=0,$$

which amounts to saying that *the two ends of any chain which is terminated by identical metals are at the same potential.*

This important proposition is a necessary consequence of the principle of the conservation of energy. If the terminal metals A and A′, of the same kind, could be kept at different potentials by intermediate contacts, then if they were joined by a conductor of the same kind, a continuous discharge would take place both in the external conductor and in the chain of intermediate metals—that is to say, a permanent flow of electricity. This transference of electricity, analogous to a succession of discharges, would, as a necessary consequence, produce calorific phenomena—that is to say, energy—which would be a realisation of perpetual motion.

If even we suppose that the disengagement of heat in certain parts of the circuit, corresponds to an absorption in other parts, there would be a transport of heat from the colder to the hotter parts, without any corresponding work. This result is incompatible with Carnot's principle, which appears as well established in science as the impossibility of perpetual motion.

190. Exception to the Law of Successive Contacts.—Electrical Batteries. —Volta found that the law of tension sometimes ceases to apply. We observe, in fact, that it does not appear necessary as long as there is no source of energy in the circuit; but if this contains sources of energy of any kind whatever—for instance, bodies which may give rise to exothermic reactions correlated to the passage of electricity—the energy furnished by these reactions, might assist in keeping up a permanent current.

Without having very accurate ideas on this point (he was prepared to find metals which did not satisfy the law), Volta had been led to divide bodies into two great classes. The first contains those which obey the law of tensions; it comprises all the metals and a certain number of solids. The second contains those which do not obey this law: it comprises most liquids and solutions. By associating bodies of the first class with those of the second, a chain can be constructed the ends of which, though formed of the same metal, present a finite difference of potential.

Experiment shows that in this case also, Volta's fundamental law is verified—that is, that the electromotive force corresponding to

each of the contacts, is a constant quantity, independent of the other bodies constituting the chain; and the difference of potential between the two ends of the chain, is the algebraical sum of all the electromotive forces due to each of the contacts.

When these two ends are joined by a conductor, no new electromotive forces of contact are introduced; a permanent flow of electricity is set up in the circuit, where it is maintained by the energy of the chemical actions. This is the principle of *electrical piles* or *batteries.*

191. CONCLUSIONS RELATIVE TO THE DISTANCE OF THE ATOMS.—When two metals are in contact, the existence of an electromotive force implies the formation of two electrical layers of opposite signs separated by a finite distance, and these layers must be localised in the two metals respectively. If we knew the electromotive force, we could determine the distance between the layers, by measuring the absolute charge on the two plates when they are separated after having been in contact; for if V denotes the electromotive force we have

$$e = \frac{S}{4\pi} \cdot \frac{V}{m}.$$

It is almost impossible to make the experiment in this form, for the charges which the plates retain depend solely on the capacity of the system at the moment the separation is effected; and this capacity is in general only a very small fraction of the original capacity, since contact cannot be broken simultaneously over the whole extent of the surface.

Sir W. Thomson, however, by a series of ingenious reasonings, has been able to show what must be the lower limit of the distance of two electrical layers.

The expression for the electrical energy of the two plates in contact is

$$W = \frac{1}{2} mV = \frac{1}{2} CV^2 = \frac{SV^2}{8\pi e},$$

and this energy represents the work necessary for separating the two plates. This conclusion may be verified in another manner. If σ is the electrical density of each of the layers, we have

$$4\pi\sigma = \frac{V}{e},$$

and the force which acts on one of the surfaces is

$$F = 2\pi\sigma^2 S = \frac{V^2 S}{8\pi e^2}.$$

In order to move the plates to a great distance apart, we must expend the work

$$T = -\int F de = -\frac{SV^2}{8\pi}\int_e^\infty \frac{de}{e^2} = \frac{SV^2}{8\pi e}.$$

The electrical energy imparted to the system at the moment of contact, is borrowed from the original potential energy of the two plates, and it cannot be supposed to be greater than that which would be available when the two metals are alloyed together.

Suppose that, with zinc and copper, the ratio of the weights of the plates is that which forms brass. Let p be the total weight, ϵ the sum of the two thicknesses, and δ the mean density of the system, we shall have

$$p = S\epsilon\delta, \qquad \text{or} \qquad S = \frac{p}{\epsilon\delta}.$$

The electrical energy may then be written

$$W = \frac{p}{8\pi\delta}\cdot\frac{V^2}{e\epsilon}.$$

This equation holds as long as the metals retain their physical properties, and for this it is evident that the total thickness of the two plates must be greater than the normal distance e of the electrical layers.

Suppose now that, without changing the total weight p, we increase simultaneously the surface of the two plates at the expense of their thickness, the energy W increases in proportion to the surface, or inversely as the thickness, so long at any rate as the thickness ϵ is greater than e. The maximum energy corresponds to the case in which $\epsilon = e$, and we have then

$$W = \frac{p}{8\pi\delta}\left(\frac{V}{e}\right)^2,$$

whence

$$e = V\sqrt{\frac{p}{8\pi\delta}\cdot\frac{1}{W}}.$$

The electrical energy must be less than that which corresponds to the alloy. As this is known to within about $\frac{1}{3}$ of its value, we may determine by the preceding equation a lower limit for the distance e of the electrical layers. If their thickness were less, the metals would have lost their properties, as they would no longer be capable of acquiring their characteristic difference of potential on contact; we may consider that the molecular constitution of these bodies would be thus modified, and that the thickness thus calculated is of the same order as the mean distance of the atoms.

Sir W. Thomson found thus, for copper and for brass,

$$e = \frac{1^{cm}}{3.10^8},$$

which corresponds to about $\frac{1}{15000}$ of the wave length of green light.

If we could separate the plates without a partial recombination of the opposite layers of electricity, each of them would be found at an extremely high potential. This potential depends on the dimensions of the plates, for the capacity of the system is proportional to the surface of contact, while the capacity of each of the plates separately is proportional to its linear dimensions. Assuming that the distance of the electrical layer is $\frac{1}{10^6}$ of a millimetre, Helmholtz has shown that if a disc of zinc 10 centimetres in radius were in contact with a disc of copper in connection with the earth, the potential of the zinc when carried to a great distance would be 39.10^6 times as great as the original potential due to contact. With Sir W. Thomson's numbers, the final potential would be 30 times as great.

192. CONTACT OF DIELECTRICS.—Volta's first law appears to apply also to the contact either of metals with dielectrics, or of dielectrics with each other. In these two cases, however, the determination of the electromotive forces of contact presents great difficulties.

A single point of contact between two conductors is sufficient to set up equilibrium of the potentials; the charges only depend then on the capacity of the system, at the moment at which their separation is completed. With bad conductors there is equilibrium at the points of actual contact only, and only these become charged with electricity. The total charge will then vary greatly with the

number of points touching. We may further add that the penetration of electricity in the dielectric still further complicates the experiments.

193. ELECTRIFICATION BY FRICTION.—In electrification by friction, the electricity seems to have no other cause than the contact of two bodies; the friction simply multiplies the number of points of contact.

When the bodies rubbed are the same, they cannot in general be electrified; very feeble traces of electricity are sometimes seen, but in that case the development of the electricity may always be attributed to a more or less visible dissymmetry between the two bodies rubbed.

194. ELECTRICAL MACHINES.—We are thus led to the conclusion that there are two modes of producing electricity—*contact* and *induction*. All electrical machines bring into play one or other of these modes, and their only object is to accumulate on a conductor the charges produced.

The general theory of these machines is very simple. Let us consider a hollow insulated conductor—that of Faraday, for instance. On the other hand, let us take the cake of an electrophorus of resin or ebonite charged with negative electricity. A metal disc held by an insulating handle, placed on the cake and connected with the ground for a moment, will become charged with positive electricity, and, if it is carried to the cylinder and made to touch the inside, an equal charge will be produced on the outer surface of the cylinder; the disc itself will be neutral on being taken out, and the same process may be repeated indefinitely. It is clear that in theory, there is no limit to the charge of the cylinder, for, whatever be the charge which it has acquired, a conductor, placed in the inside and put in communication with it, can retain no electricity. This, reduced to its simplest expression, is the mode of action of machines based on induction.

Let us now suppose that instead of the cake of the electrophorus, we have a disc of copper in connection with the earth, and that we touch it with a zinc disc held by an insulating handle; from Volta's law, the zinc disc would be charged with positive electricity; this charge, as in the preceding case, could be transferred to the cylinder, and the experiment could be repeated for an indefinite number of times.

This would also be the case if we took a piece of cloth, or of caoutchouc, instead of a copper disc, and glass disc instead of one of zinc. The insulating nature of the bodies employed must be

taken into account; contact between the cloth and the different parts of the surface of the glass, only takes place by friction. When the glass charged with positive electricity is taken to the inside of the cylinder, it would cause an equal layer of electricity, but of the opposite kind, to be formed on the inside of the cylinder; mere contact, however, with a point of the inner surface would not be sufficient for neutralisation of the two charges. This result would be obtained if the inner surface of the cylinder were charged with points: equilibrium would only be established when the density at the end of each of the points was null—that is, when the electricity which escaped had neutralised the charge of the glass disc. This is the arrangement ordinarily used in all frictional machines.

195. Essential Parts of Electrical Machines.—It will be seen that in all cases the electrical machine is reduced to three essential parts—one which develops electricity, another which transmits it, and a third which receives it: *a producer*, *a carrier*, *and a receiver*. The potential energy imparted to the collector is furnished by the mechanical work performed when the *carrier* is moved in a direction opposed to that of the electrical forces, from the *producer* charged with the contrary electricity which attracts it, to the *receiver* charged with the same electricity which repels it.

In frictional machines, the receiver takes the same quantity of electricity at each operation, and its charge increases in arithmetical progression.

Induction machines may be so arranged that the charge increases in geometrical progression: the two machines must be coupled up in such a manner that the two inductors develop electricity of opposite kinds, and that each of them is in metallic connection with the receiver of the other system. At each operation, the charge of the inducer increases at the same time as that of the receiver, with which it is connected, and induces a greater charge in the receiver at the next operation. This arrangement is made use of in Varley's machine, and in some of the very ingenious apparatus invented by Sir W. Thomson, such as the *replenisher* and the self-acting reciprocal condenser. Holtz's machine depends on the same principle, but is somewhat more complicated.

196. Limit of the Charge.—Although theory assigns no limit to the charge of the *receiver*, a practical limit is soon attained, either through the losses by air or by the supports, or by the fact of discharges which take place in the form of sparks between the receiver and the other parts of the machine or the adjacent conductors.

In the latter case, the limit only depends on the shape of the

machine and on its position in reference to the adjacent conductors; in the former case, it depends on atmospheric conditions and on the rapidity with which the operations succeed each other.

With addition machines, the limit is attained when, at any time, the gain is equal to the losses, and this limit would always have a finite value.

In the case of multiplication machines, certain conditions must be fulfilled if the charge of the receiver is to retain a finite value. Let C and C′ be the capacities of the two receivers, V and V′ the potentials in absolute values, c and c' the capacities of the carriers, and n and n' the number of operations performed in unit time.

The loss by air of an electrified conductor is sensibly proportional to the charge, or to the potential, for very feeble charges.

If m and m' are the coefficients of the proportionality relative to the two conductors, the losses of the charge for unit time might be represented by $m\mathrm{V}$ and $m'\mathrm{V}'$.

During an infinitely small time dt, the increase of charge of the receiver C is equal to the excess of the electricity which it receives, over that which it loses; we shall have then

$$\mathrm{C}d\mathrm{V} = (n'c'\mathrm{V}' - m\mathrm{V})\,dt,$$

or

$$\mathrm{C}\frac{d\mathrm{V}}{dt} = n'c'\mathrm{V}' - m\mathrm{V}.$$

The other receiver will give, in like manner,

$$\mathrm{C}'\frac{d\mathrm{V}'}{dt} = nc\mathrm{V} - m'\mathrm{V}'.$$

Solving these simultaneous differential equations, we could calculate the values reached by the potentials V and V′ at the end of a given time, starting from given initial values; but the results thus obtained would only hold within the limits in which Coulomb's law may be admitted. We know that for somewhat larger charges the loss takes place according to a more rapid law.

These equations give the conditions necessary for the charge to go on increasing; for this it is sufficient if the differentials of the potential are positive, which gives

$$n'c'\mathrm{V}' - m\mathrm{V} > 0,$$
$$nc\mathrm{V} - m'\mathrm{V}' > 0;$$

from this is deduced

$$\frac{m}{n'c} < \frac{V'}{V} < \frac{nc}{m'},$$

or

$$nn'cc' > mm'.$$

If this latter condition is realised at the outset (and it evidently does not depend on the original electrification), the charge of the machine will go on increasing. If the inequality is kept up notwithstanding the increase of the coefficients m and m', the charge will have no other limit than that which is determined by the production of sparks. If the preceding inequality were in the contrary direction, the charge would go on diminishing, and would rapidly become null.

When the apparatus is symmetrical, the condition for the increase of the charge is simply

$$nc > m.$$

The preceding calculation applies particularly to the arrangement in which the collector of one machine is in metallic connection with the inductor of the other; but the same method of reasoning, slightly modified in details, would also apply to all multipliers of electricity which act by reciprocal induction.

197. YIELD OF MACHINES.—Whatever, moreover, may be the external cause limiting the charge, it will be seen that all these machines act like true *sources*—that is to say, as systems which by the play of their own organs can maintain a conductor at a constant potential, or maintain a certain difference of potential between two conductors.

This result is obtained when the quantity of electricity brought to the conductor is at every instant equal to that taken away from it, either by loss from contact with air, or by discharges between the collector and the earth. The *yield* of the machine is the quantity of electricity put in motion in each unit of time. It is clear that for addition machines, the yield, other things being equal, is proportional to the capacity of the carrier and to the number of operations performed in each unit of time. If, as with plate machines, the carrier acts continuously, the yield is proportional to the velocity.

The phenomena in multiplication machines are not quite so simple; but experiment shows that the yield is sensibly proportional to the velocity, although as a general rule it increases a little more rapidly.

PART II.—ELECTRICAL CURRENTS.

CHAPTER I.

PROPAGATION OF ELECTRICITY IN THE PERMANENT STATE.

198. PERMANENT CONDITION.—When two insulated conductors, at different potentials V and V′, are put in metallic connection, equilibrium can no longer exist—positive electricity flows from the body at the higher towards the body at the lower potential; a *flow of electricity*, an *electrical current* is produced. If the charges on the two bodies are limited, equilibrium is established after a time, which is generally very short, and which depends on the nature and the dimensions of the intermediate conductor; the current is then *variable* with the time. But if by any means the two conductors are kept at a constant difference of potential, a permanent state is established, and the intermediate conductor becomes the seat of a *constant current.*

199. ANALOGY WITH THERMAL PHENOMENA.—The analogy of these phenomena with those of the propagation of heat between surfaces, at constant temperatures in a conducting medium, is obvious, and this analogy is expressed by identical laws in the two cases.

We have seen, reminding the reader of the principles of Fourier's theorem (70), that if, in a medium which is a conductor of heat, we take two near isothermal surfaces at the temperatures t and $t+dt$, the flow of heat $d\mathrm{Q}$, which in unit time traverses an element of surface $d\mathrm{S}$, is perpendicular to the element, proportional to the difference of temperature dt of the two surfaces, and inversely as their distance dn, and is thus expressed—

$$d\mathrm{Q} = -k d\mathrm{S} \frac{dt}{dn},$$

k being the coefficient of conductivity for heat; the sign – signifies that the flow of heat is in the direction in which the temperatures decrease. The expression for the flow is the same across an element $d\mathrm{S}'$ of any given surface S', isothermal or not; it is proportional to the partial differential $\frac{\partial t}{\partial n'}$ of the temperature in reference to the perpendicular n' to the surface S', and we have

$$d\mathrm{Q}' = -kd\mathrm{S}'\frac{\partial t}{\partial n'}.$$

200. OHM'S THEORY.—Ohm transferred Fourier's method of reasoning to the study of the propagation of electricity. He assumes that all points of a conductor in equilibrium are in the same electrical condition, at the same *tension*. When there is no equilibrium, interchanges of electricity take place; the tension at every point is generally a function of the time and of the co-ordinates; but if any extraneous cause maintains a constant difference between the tensions of the different parts of the conductor, a stationary condition is established in the system, after a shorter or longer time, in which the tension at each point becomes independent of the time.

Ohm assumes, further, that between two molecules whose tensions are U and U', an exchange of electricity is produced in unit time, proportional to the difference of tensions and to a function of the distance, such that the adjacent molecules have a preponderating influence.

This hypothesis is identical with that of Fourier (70). Without its being necessary to repeat the reasoning, it follows that the exchanges of electricity take place at right angles to the surfaces of equal tension, or to the surfaces of electrical level relative to this new property. The flow of electricity $d\mathrm{Q}$, which traverses an element $d\mathrm{S}$, of an equipotential surface in unit time, is proportional to the differential of the tension in respect of the perpendicular to this surface, and is expressed by

$$d\mathrm{Q} = -cd\mathrm{S}\frac{d\mathrm{U}}{dn},$$

the coefficient c depending on the nature of the medium, and may be called the *coefficient of electrical conductivity*. The differential $-\frac{d\mathrm{U}}{dn}$ is the *electromotive force* at the point in question. It will be seen that through an element $d\mathrm{S}'$, of any given surface, there is a flow

of electricity which is proportional to the electromotive force along the normal to this surface, or to the differential coefficient $\frac{\partial U}{\partial n'}$.

On Ohm's theory, as well as on that of Fourier, it is assumed that the direction of the flow is constantly parallel to the direction of the force which acts upon it, and is therefore independent of its previous condition. This hypothesis is incompatible with the notion of inertia, and therefore with the materiality of that which constitutes the flow.

Ohm regarded tension as a particular condition in virtue of which electricity tends to escape, and, when the tensions in any medium are variable, electricity flows from points at a high towards points at a lower tension. We see at once the analogies of this function with the potential, for the tension is also constant in a conductor in equilibrium. In some of his memoirs Ohm appeared to establish too close a relation between electrical density and tension; but he also points out that the tension at a point, even in the variable state, could be determined by connecting it with an insulated electroscope, and the quantity, which is thus measured, is nothing else than the potential.

201. KIRCHHOFF'S HYPOTHESIS.—To express Ohm's theory with our present views respecting electricity, we must assume with Kirchhoff that tension and potential are two identical functions.

It follows from this hypothesis that, in a system of conductors connected together, but not in electrical equilibrium, the flow of electricity at each point is proportional to the differential of the potential in reference to the normal to the equipotential surface, or, in other words, is proportional to the electrical force exerted at this point—that is to say, to the resultant of the actions of all the masses of the system in their condition for the time being.

Or, more simply, *the flow of electricity is parallel and proportional to the flow of force.*

202. SUPERPOSITION OF PERMANENT STATES.—In another form it may be said that the hypothesis amounts to the assumption that the superposition of two states of electrical equilibrium is itself a new state of equilibrium, in which the flow across an element of surface is equal to the sum of the flows relative to the two original states.

For consider two states in which the flows across an element of surface dS are AdS and A'dS; the perpendicular components of the force on this element are $-\frac{\partial V}{\partial n}d$S and $-\frac{\partial V'}{\partial n}d$S. By superposing

the two states, the potentials at each point add together, and the perpendicular force upon the element dS becomes

$$-\left(\frac{\partial V}{\partial n}+\frac{\partial V'}{\partial n}\right)dS.$$

By hypothesis, the flow across the element dS has become

$$AdS+A'dS.$$

If we suppose the initial conditions to be identical, the flow becomes $2AdS$, and the force is thus doubled; the flow of electricity is therefore proportional to the flow of force.

If the state is permanent—that is, if the potential is unchanged at each point—the flow across any surface-element is constant. If the state is variable, the potentials vary with the time, and the flow d^2Q across an element of surface during the time dt is

$$d^2Q=-cdS\frac{\partial V}{\partial n}dt.$$

203. In the Permanent State the Density in the Interior of a Conductor is Null.—The flow of electricity cannot accumulate in the interior of a closed surface, without modifying the potential, just as an increase in the flow of heat would produce there a rise of temperature.

If the permanent state has been attained for a system of conductors, the total flow which each element receives is zero. If the algebraical sum of the flows which enter a volume-element $dxdydz$ is equal to zero, we find again Laplace's equation:

$$\Delta V=0.$$

ρ being the cubical density of electricity at the point in question, we have always

$$\Delta V=-4\pi\rho,$$

from which

$$\rho=0.$$

The law of the proportionality of the flow of electricity to the flow of force leads directly to the same conclusion. For, if the state

is permanent, the flow of electricity for a volume-element is zero: the flow of force which is proportional to it is zero also; but the latter is equal to $4\pi m$, m being the mass in the volume in question; hence $m = 0$. Thus, when a system of conductors has attained a permanent state, the electrical density is zero at all points of the conductor; the electrical masses which produce the potential V, and whose action determines the current, are therefore entirely on the surface of the conductors. These masses are not in equilibrium of themselves, and they produce at each point the electromotive force of the current.

It follows from this that the flow, whatever it may be, if it has a real existence, is not a flow of free electricity; on the hypothesis of two fluids, we must assume that at every instant there is the same quantity of the two electricities in each volume-element in the interior of the conductor, and that these move in two equal currents in opposite directions. On the hypothesis of a single fluid, each element must be looked upon as containing at each instant the normal quantity of electricity, while we still assume that this may be either wholly or partially displaced.

204. LINEAR CONDUCTORS.—OHM'S LAW.—Imagine a cylindrical wire, very long as compared with its diameter, placed in a perfectly insulating medium, and let us suppose that the permanent state has been attained.

If there is no loss of electricity, the flow of electricity is parallel at each point to the generating lines of the cylinder; the equipotential surfaces are therefore planes perpendicular to the axis of the wire. The flow of electricity across any given section in unit time is the same throughout the whole length; let us call this flow the *intensity* or *strength of the current*, or more simply the *current*, and denote it by i. Let V be the potential at the point P at a distance x from a fixed plane perpendicular to the wire, and let S be the section of the wire.

The potential is simply a function of x, and the expression for the current is

$$i = -c\mathrm{S}\frac{d\mathrm{V}}{dx}.$$

As this flow is independent of x, we have

$$\frac{d\mathrm{V}}{dx} = a, \qquad \text{and} \qquad \mathrm{V} = ax + b,$$

where a and b are constants to be determined.

Let V_1 and V_2 be the values of the potential at two points A and B (Fig 49) at a distance l from each other, and let the point A be the origin of the co-ordinates, we have

$$(1) \qquad V = -\frac{V_1 - V_2}{l} x + V_1,$$

and, consequently,

$$(2) \qquad i = cS\frac{V_1 - V_2}{l} = \frac{V_1 - V_2}{\frac{l}{cS}}.$$

The quotient $\frac{l}{cS} = r$ is called the *resistance* of the wire between the two points A and B, and the inverse of this resistance $\frac{cS}{l}$ is the *conductivity* of this same wire.

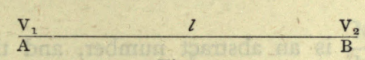

Fig. 49.

Equations (1) and (2) show that:

1st. *The potential decreases in arithmetical progression along the wire, in the direction of the current;*

2nd. *The current between the two points A and B is equal to the quotient of the difference of potential of these two points by the resistance of the intermediate wire.*

These two statements form *Ohm's law.*

It may be noticed that the distribution of potential, and the flow of electricity in the case we are considering, are identical with the distribution of temperatures, and with the flow of heat in a homogeneous wall bounded by two parallel planes kept respectively at constant temperatures.

205. THE RESISTANCE OF A CONDUCTOR IS THE INVERSE OF A VELOCITY.—The quantity r, which we have called the resistance of the conductor, has the value $\frac{l}{cS}$; it is proportional to the length of the conductor, is inversely as its section, and of the coefficient of conductivity of the medium.

The ratio $\frac{1}{c}$ represents the resistance of a cube equal to unity parallel to one edge; it may be called the *specific resistance* of the conductor.

The resistance of a conductor is a magnitude of the same kind as the inverse of a velocity in mechanics.

For we have

$$r = \frac{V_1 - V_2}{i}.$$

The difference of potential $V_1 - V_2$ is equal to the quotient of an electrical mass M by a length a; the strength of a current, or the flow during unit time, is equal to the ratio of the quantity of electricity M′, which flows in the time t, to the corresponding time. We have then

$$r = \frac{\frac{M}{a}}{\frac{M'}{t}} = \frac{M}{M'} \cdot \frac{1}{\frac{a}{t}}.$$

The quotient $\frac{M}{M'}$ is an abstract number, and the ratio $\frac{a}{t}$ is a velocity. The resistance r is therefore the inverse of a velocity.

206. We may, indeed, imagine an experiment in which this velocity would have a physical meaning.

Consider a sphere, of radius R, charged with a mass of electricity M, and let us suppose this sphere connected with the earth by a conductor of resistance r.

The potential of this sphere is equal to $\frac{M}{R}$; it diminishes as soon as it is connected with the earth; but if the sphere contracts at the same time as the charge diminishes, it may happen that the potential remains constant.

This condition being realised, then if dM is the loss of charge of the sphere, and dR the diminution of the radius in the time dt,

$$V = \frac{M}{R} = \frac{M - dM}{R - dR} = \frac{dM}{dR};$$

but from Ohm's law,

$$dM = \frac{V}{r} dt.$$

In order that the potential may be constant, $d\mathrm{R}$ must also be proportional to the time; let us put

$$d\mathrm{R} = u dt.$$

We have then

$$\mathrm{V} = \frac{\frac{\mathrm{V}}{r} dt}{u dt} = \frac{\mathrm{V}}{ru}.$$

and therefore

$$r = \frac{1}{u}.$$

Thus the resistance r of a given conductor is the inverse of the velocity u with which the radius of a sphere must decrease for its potential to remain constant, notwithstanding the loss of electricity, when it is connected with the earth by the conductor in question.

207. ANY GIVEN LINEAR CONDUCTORS.—We have supposed the conductors to be rectilinear, but the same reasoning evidently applies to linear conductors bent in any manner whatever, the flow of electricity being perpendicular at each point to the cross section of the conductor.

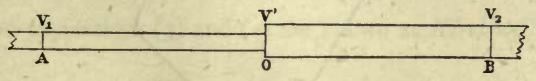

Fig. 50.

If the circuit consists of two or more cylindrical portions of different kinds and sections joined end to end, these various parts may be considered separately.

If V_1 and V_2 are the potentials at the points A and B (Fig. 50), the first in a conductor of section S, and whose coefficient of conductivity is c, and the second belonging to a conductor in which these quantities are S′ and c'. Let V′ be the point of contact O of the two cylinders, at distances l and l' from A and B respectively, and let us for the present disregard the electromotive force of contact of the two conductors, to which we shall subsequently return. On each side of the point O we have

$$i = \frac{\mathrm{V}_1 - \mathrm{V}'}{\frac{l}{c\mathrm{S}}} = \frac{\mathrm{V}' - \mathrm{V}_2}{\frac{l'}{c'\mathrm{S}'}} = \frac{\mathrm{V}_1 - \mathrm{V}_2}{\frac{l}{c\mathrm{S}} + \frac{l'}{c'\mathrm{S}'}} = \frac{\mathrm{V}_1 - \mathrm{V}_2}{r + r'}.$$

The current is therefore inversely as the sum of the resistances

of the two conductors between the points A and B. This is obviously a general relation.

Thus the resistance of a series of successive cylindrical conductors is the sum of the resistances of all the conductors.

In conclusion, let us take the case of a conductor of any given shape terminated at its ends by equipotential surfaces kept at potentials V_1 and V_2; the current is proportional to the difference $V_1 - V_2$ of the potentials, and the number by which this difference must be divided to give the strength of the current represents the resistance of the conductor. The number thus obtained is the resistance of the cylindrical conductor, which for the same difference of potential would give the same current.

208. KIRCHHOFF'S LAWS.—Let us suppose linear conductors, of various materials and different sections, to be joined to each other in a complicated manner, the division of the current among these various conductors must satisfy the two following conditions, which follow directly from Ohm's law.

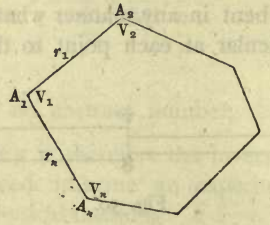

Fig. 51.

1st. *If several conductors terminate at the same point, the sum of the currents, counted from this point, is zero.*

For, since there can be no accumulation of electricity at the point in question, the quantity of electricity brought by one set of conductors must be equal to that which passes away by the others in the same time; so that if we give the positive sign to the currents proceeding towards the point, and the negative sign to those which pass away, we must have

$$\sum i = 0. \tag{3}$$

2nd. *If several conductors form a closed polygon, the sum of the products of the resistance of each conductor, by the current which traverses it, is zero.*

Imagine a series of conductors of resistances $r_1, r_2, r_3, \ldots\ldots r_n$, which form the successive sides of a closed polygon (Fig. 51); let

$V_1, V_2 \ldots\ldots V_n$, be the potentials at the summits $A_1, A_2 \ldots\ldots A_n$, and let $i_1, i_2 \ldots\ldots i_n$ be the currents reckoned positively when the circuit is traversed in a certain direction. These strengths are not equal, for at the various summits there may be other conductors which bring or carry away currents.

We shall have successively :—

For the first conductor $\quad i_1 r_1 = V_1 - V_2,$
For the second conductor $\quad i_2 r_2 = V_2 - V_3,$
$\vdots \qquad\qquad \vdots$
For the n^{th} conductor $\quad i_n r_n = V_n - V_1.$

Adding these equations together, all the potentials disappear, and we have finally

$$i_1 r_1 + i_2 r_2 + \ldots\ldots + i_n r_n = 0,$$

from which

$$\sum ir = 0. \tag{4}$$

The two expressions (3) and (4) are known as *Kirchhoff's laws.*

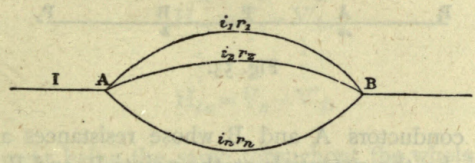

Fig. 52.

209. RESISTANCE OF A MULTIPLE CONDUCTOR.—As an application of these theorems, let us consider the case in which the circuit divides into multiple arcs, between two points A and B (Fig. 52). Let I be the current in the undivided part in front of A and beyond B, $r_1, r_2 \ldots r_n$ the resistances of the conductors, $i_1, i_2 \ldots i_n$ the respective currents, and lastly let R be the resistance of the single circuit which would be equivalent to the multiple circuit between the same two points. We shall have

$$I = i_1 + i_2 + i_3 \ldots\ldots + i_n = \sum i,$$
$$i_1 r_1 = i_2 r_2 = i_3 r_3 = i_n r_n = IR.$$

The second equation may be written

$$\frac{i_1}{\frac{1}{r_1}} = \frac{i_2}{\frac{1}{r_2}} = \ldots\ldots = \frac{i_n}{\frac{1}{r_n}} = \frac{I}{\frac{1}{R}},$$

and we deduce from it

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots\ldots + \frac{1}{r_n} = \Sigma\frac{1}{r}.$$

Thus the inverse of the resistance of a number of conductors terminating at the same points, is the sum of the inverses of the resistances of the several separate conductors; in other words, the conductivity of a multiple conductor is the sum of the conductivities of the several conductors of which it is made up, which indeed is evident.

210. HETEROGENEOUS LINEAR CONDUCTORS.—The existence of electromotive forces of contact between metals slightly modifies Ohm's law. Consider two points P_1 and P_2 (Fig. 53), separated by

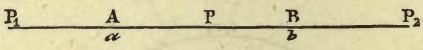

Fig. 53.

two different conductors A and B whose resistances are a and b. Let V_1 and V_2 be the potentials at the points P_1 and P_2, V_a and V_b the potentials at the point P on each side of the surface of separation of the metals. The current from P_1 to P_2 is

$$I = \frac{V_1 - V_a}{a} = \frac{V_b - V_2}{b} = \frac{V_1 - V_2 + V_b - V_a}{a + b}.$$

Let H_{ab} be the sudden rise which the potential experiences between the metals A and B going with the current—that is to say, the electromotive force of contact $V_b - V_a = B|A$ of the conductors B and A, we get

$$I = \frac{V_1 - V_2 + H_{ab}}{a + b}.$$

Let us now suppose that a closed circuit is composed of separate conductors A, B, C..... L (Fig. 54) comprising bodies of Volta's second class—that is to say, that the chain of conductors does not obey the law of tensions; the circuit will be traversed by a permanent current.

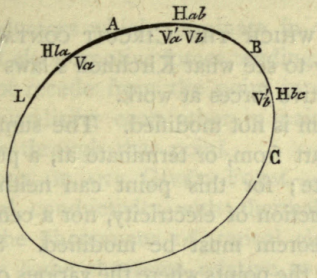

Fig. 54.

Let $r_a, r_b, r_c \ldots\ldots r_l$, be the resistances of the different conductors, V_a and V'_a, V_b and $V'_b \ldots\ldots$ the successive potentials at the ends of each going in the direction of the current, and in like manner let H_{ab}, $H_{bc} \ldots\ldots H_{la}$ be the successive electromotive forces of contact; we have

$$\begin{aligned} H_{ab} &= V_b - V'_a, \\ H_{bc} &= V_c - V'_b, \\ &\vdots \quad \vdots \quad \vdots \\ H_{la} &= V_a - V'_l, \end{aligned}$$

The current being the same throughout the whole extent of the circuit, we have also

$$I = \frac{V_a - V'_a}{r_a} = \frac{V_b - V'_b}{r_b} = \ldots\ldots = \frac{V_l - V'_l}{r_l},$$

from which, reducing, we get

$$I = \frac{(V_b - V'_a) + (V_c - V'_b) + \ldots + (V_a - V'_l}{r_a + r_b + r_c + \ldots\ldots + r_l} = \frac{H_{ab} + H_{bc} + \ldots + H_{la}}{r_a + r_b + r_c + \ldots\ldots + r_l}.$$

The numerator of this fraction represents the algebraical sum $\sum H$ of the electromotive forces of contact in the chain of conductors; it is the electromotive force E of the circuit. The

denominator is the sum of the resistances, or the total resistance of the circuit. We have thus

$$I = \frac{\Sigma H}{R} = \frac{E}{R}. \tag{5}$$

211. CASE IN WHICH THE CIRCUIT CONTAINS ELECTROMOTIVE FORCES.—It is easy to see what Kirchhoff's laws (208) become, when there are electromotive forces at work.

The first theorem is not modified. The sum of the quantities of electricity which start from, or terminate at, a point is always zero in the permanent state; for this point can neither be an unlimited centre for the production of electricity, nor a centre of absorption.

The second theorem must be modified. Suppose that in the preceding circuit, at the points where the various conductors terminate, the metals change, or that these places are also points where the summits of other conductors branch off. The current is not the same throughout the whole circuit; let $i_a, i_b \ldots . i_l$ be the different values of the current in the different conductors between any two successive points of contact or of division. From Ohm's law we shall have

$$\begin{aligned} V_a - V'_a &= r_a i_a, \\ V_b - V'_b &= r_b i_b, \\ &\cdots\cdots\cdots\cdots \\ V_l - V'_l &= r_l i_l, \end{aligned}$$

and consequently

$$\begin{aligned} r_a i_a + r_b i_b + \ldots + r_l i_l &= (V_a - V'_a) + (V_b - V'_b) + \ldots + (V_l - V'_l) \\ &= (V_b - V'_a) + (V_c - V'_b) + \ldots + (V_a - V'_l) \\ &= H_{ab} + H_{bc} + \ldots + H_{la} = \Sigma H = E, \end{aligned}$$

or

$$E = \Sigma ri. \tag{6}$$

Thus, in a closed circuit, *the sum of the products of the resistance of each conductor by the strength of the corresponding current is equal to the algebraical sum of the electromotive forces of the circuit.*

This sum is zero if the circuit is made up of conductors of the same kind, or of metals at the same temperature, for the latter obey the law of tensions.

The two relations (5) and (6) give all the equations necessary

for determining the currents in the various branches of the circuit. Any modification in the resistances, or in the electromotive forces which produces no change in the equations, will obviously be without influence on the currents. For instance:

1st. The resistance of a branch in which the current is null may be modified at will.

2nd. In all conductors which terminate in the same point, we may introduce equal electromotive forces tending to produce currents which all approach or recede from the point in question, these electromotive forces neutralising each other in pairs in all the closed contours which pass through that point.

212. CONDUCTORS OF ANY GIVEN FORM.—ELECTRODES.—The analogy of electrical conductivity with thermal conductivity, and of this latter with the phenomena of statical electricity, enables us to establish directly some theorems relative to the propagation of electricity.

Let us consider, in the first place, a single isotropic unlimited medium. Let us suppose that, for the three different orders of phenomena, the temperature on the one hand and the potential on the other are kept constant on different closed surfaces S_1, S_2, ... S_n, and that on each of the surfaces the temperatures and the potentials t_1 and V_1, t_2 and V_2..... are represented by the same numbers, or by proportional numbers. In the thermal problem, these surfaces will represent sources of heat; in the problem of statical electricity, the conductors; in the problem of the propagation of electricity, they are called the *electrodes*.

The temperature and potential of any point of the medium are functions of the co-ordinates defined by the condition that these functions acquire determinate values on the bounding surfaces, and in the interval of these surfaces satisfy the condition

$$\Delta t = 0, \quad \text{or} \quad \Delta V = 0.$$

The temperature and the potential at every point will therefore have values which are either equal or are in a constant ratio. The isothermal and the equipotential surfaces will be identical throughout the whole extent of the medium, and therefore the tubes of flow are identical.

Across an element $d\mathrm{S}$ of any given surface, the flow of heat (70) is $-kd\mathrm{S}\frac{\partial t}{\partial n}$, the flow of electrostatic force is $-d\mathrm{S}\frac{\partial V}{\partial n}$, and the electrical current is $-cd\mathrm{S}\frac{\partial V}{\partial n}$, k and c being the coefficients of thermal

and electrical conductivity. These three flows are therefore proportional to each other.

According to this, whenever a problem of the uniform propagation of heat or of statical electricity has been solved, the corresponding problem of the propagation of electricity in the permanent state, will be found to be solved in the same way.

As a particular case, we have seen that if we assign the potentials $V_1, V_2 \ldots, V_n$ to fixed surfaces $S_1, S_2 \ldots, S_n$—that is to say, the potentials of different conductors in air—the potential at every point of the medium is defined, and the problem of equilibrium has only one solution. In like manner, if the dielectric medium is replaced by a conductor, and the potentials are kept constant on the same surfaces $S_1, S_2 \ldots S_n$, the flow of electricity is determined for each point, and there is only a single state of equilibrium.

213. Suppose now that two dielectrics, whose inductive capacities are μ_1 and μ_2, are separated by a surface S. The flow of force is not maintained on each side of the surface, but the flow of induction is maintained, and counting the normal n to the surface in the same direction for the two media, we have (121)

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$$\mu_1 \frac{\partial V_1}{\partial n} = \mu_2 \frac{\partial V_2}{\partial n}.$$

It is the condition of continuity on the surface. If we replace the dielectrics by conductors whose coefficients of conductivity are c_1 and c_2, the flow of electricity is then the same on each side of the surface S, which gives

$$c_1 \frac{\partial V_1}{\partial n} = c_2 \frac{\partial V_2}{\partial n}.$$

It will be seen from this that if, in a problem of electrostatics containing dielectrics whose inductive capacities are $\mu_1, \mu_2, \ldots \mu_3$, we replace the dielectrics by conductors whose coefficients of conductivity are respectively proportional to the corresponding inductive capacities, such, that is to say, that we have

$$\frac{\mu_1}{c_1} = \frac{\mu_2}{c_2} = \ldots\ldots = \frac{\mu_n}{c_n},$$

the flow of electricity at each point will be proportional to the flow of induction of the correlated electrostatic system.

Thus, all electrostatical problems which have been solved for a system of dielectrics, furnish also the solution of the corresponding problems in the propagation of electricity. Such, for instance, are the following cases :—

Concentric spheres (77).

Concentric cylinders (80), or eccentric cylinders one of which is inside the other (136).

Parallel planes (81).

Closed condensers of constant thickness (79).

Successive concentric cylinders formed of different media (164).

214. HETEROGENEOUS CONDUCTORS.—We have also seen (167) that, by comparing a dielectric to a medium whose specific inductive capacity is μ_2, and in which is disseminated little spheres of the specific inductive capacity μ_1, the medium thus constituted behaves like a homogeneous dielectric, the specific inductive capacity of which would be represented by the expression

$$\mu = \mu_2 \frac{\mu_1 + 2\mu_2 - 2h(\mu_2 - \mu_1)}{\mu_1 + 2\mu_2 + h(\mu_2 - \mu_1)},$$

in which h represents the ratio of the sum of the volumes of the small spheres to the total volume of the space in which they are disseminated.

In analogous conditions, the mean specific conductivity of a medium of conductivity c_2, containing small spheres of conductivity c_1, is expressed by the formula

$$c = c_2 \frac{c_1 + 2c_2 - 2h(c_2 - c_1)}{c_1 + 2c_2 + h(c_2 - c_1)}.$$

If the ratio $\frac{c_1}{c_2}$ of the conductivities of the spheres and the surrounding media is very great, the formula reduces to

$$c = c_2 \frac{1 + 2h}{1 - h}.$$

In like manner, the problem of 150 will give the conductivity of a system formed of three different media separated by two parallel planes.

215. Anisotropic Conductors.—We have hitherto only considered the case of isotropic bodies—that is to say, bodies which have the same properties in all directions. If the medium is anisotropic, but homogeneous like crystallized bodies, the physical phenomena depend on the direction in which they are regarded.

The expansion, for instance, may be very unequal. There are then three principal directions, rectangular to each other, and such that the expansion of an infinitely thin cylinder considered in the medium parallel to one of the principal directions takes place along the axis of the cylinder. Each of these directions is defined by a particular coefficient, which gives for the medium three principal coefficients of expansion, l, l' and l''. If we suppose in the medium an infinitely thin cylinder in any given direction, making angles with the principal directions, the cosines of which are α, α' and α'', this cylinder turns at the same time that it dilates, but remains rectilinear if the medium is homogeneous. The expansion parallel to the axis of the cylinder is equal to the sum of the projections on this axis of the three principal expansions, and the value of the coefficient L relative to this direction is

$$L = l\alpha + l'\alpha' + l''\alpha''.$$

The same considerations apply to the propagation of heat, to the propagation of electricity, and to electrostatic induction.

In an anisotropic medium, the flow of heat at a point is no longer perpendicular to the corresponding isothermal surface; but, just as in the case of the expansion, and generally for all properties which are linear functions of the causes on which they depend, there are again three rectangular directions along which the flow of heat is perpendicular to the isothermal surfaces, and to which correspond the three principal coefficients of conductivity k, k' and k''.

Across an element of surface dS, the perpendicular to which makes angles with the axes the cosines of which are α, α' and α'', the flow of heat is equal to the sum of the flows which correspond to the projections αdS, $\alpha' d$S, $\alpha'' d$S of the element perpendicular to the three principal axes; taking, then, these three directions for axes of the co-ordinates,

$$dQ = -dS\left[k\alpha\frac{\partial t}{\partial x} + k'\alpha'\frac{\partial t}{\partial y} + k''\alpha''\frac{\partial t}{\partial z}\right].$$

In like manner again, if c, c', c'' are the coefficients of electrical

conductivity of the medium along the principal axes, the flow of electricity $d\mathrm{M}$ across an element of surface $d\mathrm{S}$ will have an analogous expression as a function of potentials. If I_n is the strength of the current for unit surface along the perpendicular to the current, we shall have

$$d\mathrm{M} = \mathrm{I}_n d\mathrm{S} = -d\mathrm{S}\left[c\alpha\frac{\partial \mathrm{V}}{\partial x} + c'\alpha'\frac{\partial \mathrm{V}}{\partial y} + c''\alpha''\frac{\partial \mathrm{V}}{\partial z}\right].$$

or

$$\mathrm{I}_n = -\left[c\alpha\frac{\partial \mathrm{V}}{\partial x} + c'\alpha'\frac{\partial \mathrm{V}}{\partial y} + c''\alpha''\frac{\partial \mathrm{V}}{\partial z}\right].$$

This latter expression is the sum of the projections on the perpendicular of the currents I, I′, I″ in the three principal directions, or the projection of the resultant current I_0.

If θ is the angle made by the perpendicular to the element $d\mathrm{S}$ with the direction of the current, which makes with the axes angles whose cosines are λ, λ', λ'', we have then

$$\mathrm{I}_n = (\alpha\mathrm{I} + \alpha'\mathrm{I}' + \alpha''\mathrm{I}'') = (\alpha\lambda + \alpha'\lambda' + \alpha''\lambda'')\mathrm{I}_0 = \mathrm{I}_0 \cos\theta,$$

and

$$\frac{\lambda}{c\dfrac{\partial \mathrm{V}}{\partial x}} = \frac{\lambda'}{c'\dfrac{\partial \mathrm{V}}{\partial y}} = \frac{\lambda''}{c''\dfrac{\partial \mathrm{V}}{\partial z}}.$$

216. We may now without difficulty extend the same kind of reasoning to phenomena of induction in anisotropic dielectrics. Here again are three principal directions of induction, such that the flow of force is perpendicular to the equipotential surfaces, and which we may characterise by the specific inductive capacities μ, μ' and μ''. If ϕ is the flow for unit surface, the flow of induction across an element $d\mathrm{S}$ is

$$\phi d\mathrm{S} = -d\mathrm{S}\left[\mu\alpha\frac{\partial \mathrm{V}}{\partial x} + \mu'\alpha'\frac{\partial \mathrm{V}}{\partial y} + \mu''\alpha''\frac{\partial \mathrm{V}}{\partial z}\right].$$

If β, β' and β'' are the cosines of the angles formed by the electrical force F at the point in question, which is not perpendicular to the equipotential surface, we may write

$$\phi = (\mu\alpha\beta + \mu'\alpha'\beta' + \mu''\alpha''\beta'')\,\mathrm{F}.$$

In this case the electrical displacement is no longer parallel to the electrical force.

217. Conductors in Two Dimensions.—The preceding considerations apply to unlimited media. It is clear that nothing is altered if we limit the conducting medium by a surface formed entirely by the lines of flow of the unlimited system. For a limited conductor placed in an insulating medium, the external surface, whatever it may be, is always parallel to the lines of flow, and therefore, if the medium is isotropic, it is perpendicular to the equipotential surfaces.

This is the case of the propagation of electricity in a thin plate, which may be regarded as a conductor of two dimensions. We may determine by experiment the locus of points which have the same potential, by the condition that no current flows through a conducting wire one end of which is in connection with a fixed point in the plane. The results obtained by experiment are in complete accordance with those deduced from Fourier's formula, and furnish a fresh confirmation of the analogy between the two orders of phenomena.

In both cases we may suppose the propagation to take place either with or without loss in the surrounding medium.

If there is no loss in the external medium, Poisson's equation for any point outside the electrodes reduces to

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0,$$

and, for any point in the outside of the plate, we have

$$\frac{\partial V}{\partial n}=0.$$

It is easily seen that this problem merges into that of the problem of equilibrium in the case of a cylindrical distribution (132 *et seq.*) We have seen there that in a plane traversed perpendicularly at points A_1, A_2, A_3..... by parallel lines of the densities λ_1, λ_2, λ_3..., the potential at a point P, at distances r_1, r_2, r_3... from these lines, has the value

$$V=\text{const}-2\sum\lambda l.r.$$

The flows of force which, in a layer of the thickness ϵ comprised

between two parallel planes, proceed from the different lines are

$$\phi_1 = 4\pi\lambda_1\epsilon,$$
$$\phi_2 = 4\pi\lambda_2\epsilon,$$
$$\ldots\ldots\ldots$$

If, in the problem of propagation, we regard the same portions of the lines as sources of electricity, as electrodes, the flows of electricity, or the strengths of the currents, are

$$I_1 = c\phi_1 = 4\pi c\lambda_1\epsilon,$$
$$I_2 = c\phi_2 = 4\pi c\lambda_2\epsilon,$$
$$\ldots\ldots\ldots\ldots\ldots;$$

from which we have

$$\lambda_1 = \frac{I_1}{4\pi c\epsilon},$$
$$\lambda_2 = \frac{I_2}{4\pi c\epsilon},$$
$$\ldots\ldots\ldots,$$

and the expression of the potential at P becomes

$$V = \text{const} - \sum\frac{Il.r}{2\pi c\epsilon} = \text{const} - \frac{1}{2\pi c\epsilon}\left[I_1 l.r_1 + I_2 l.r_2 + \ldots\ldots\right].$$

A particularly interesting case is that of two electrodes A_1 and A_2 furnishing equal flows of opposite signs. In this case

$$I_1 + I_2 = 0,$$

and

$$V = \text{const} - \frac{I}{2\pi c\epsilon} l.\frac{r_1}{r_2}.$$

The lines of flow are segments of circles passing through the two points A_1 and A_2 (Fig. 30); the equipotential lines are circumferences having their centres on the line $A_1 A_2$ and such that these two points are conjugate in reference to each of them. From the remark made above, it is clear that the problem will remain the same

if, instead of an unlimited plate, we consider a circular plate having two electrodes on its circumference, or again, any plate comprised between two circular segments passing through the points A_1 and A_2.

218. Resistance of a Conductor of any given Form.—Whatever be the conductor, it may be always supposed to be divided by two series of surfaces parallel to the lines of flow into infinitely slender tubes, each of which is a tube of flow. Each of these tubes may itself be compared to a conducting wire of varying section, the resistance of which is at each point inversely as the section. The total resistance can be deduced from the resistance of this complex, by the ordinary laws of multiple conductors; the reciprocal of the total resistance, or the conductivity, will be the sum of the reciprocals of the resistances of all these tubes.

The calculation will in general be very complicated; but if the value of the potential on the two electrodes is known, as well as the corresponding flow of electricity, it is easy to determine the total resistance of the conducting medium by Ohm's formula (207).

Let us take as an example the case of two electrodes A_1 and A_2 in an unlimited medium. We may suppose these electrodes to be small spheres of radius ρ. Let V_1 and V_2 be their potentials, and I the absolute value of the flow of electricity corresponding to each of them. If the radius ρ can be neglected in comparison with the distance $A_1 A_2$, we may assume that the potential close to each of the electrodes is inversely as the distance r, and is represented by $\frac{b}{r}$, which, on the spheres themselves, will give

$$V_1 = -V_2 = \frac{b}{\rho}.$$

The current is then

$$I = c\int F_n dS = -4\pi\rho^2 c\frac{dV}{dn}.$$

As, on the surface of the sphere, we have

$$\frac{dV}{dn} = -\frac{b}{r^2} = -\frac{b}{\rho^2},$$

we get

$$I = 4\pi cb = 4\pi cV_1\rho = -4\pi cV_2\rho,$$

or

$$I = 2\pi c\rho\,(V_1 - V_2).$$

According to this, the total resistance R of the medium is expressed by

$$R = \frac{I}{2\pi c\rho}.$$

The same reasoning would apply to the case of a medium unlimited on one side, and bounded on the other by a plane on which two hemispherical electrodes A_1 and A_2 are placed. The resistance would then be the double of the preceding, and we should have

$$R = \frac{I}{\pi c\rho}.$$

It is remarkable that the resistance is independent of the distance of the two electrodes, and only depends on their dimensions and on the conductivity of the medium. This case may be regarded as corresponding to that of the earth when two points of the soil are connected with electrodes kept at potentials of equal values and opposite signs.

219. DISTRIBUTION OF ELECTRICITY ON LINEAR CONDUCTORS.—When the state is permanent, as the density is zero in the interior of the conductor (**203**), the potential is simply due to the electricity which exists on the surface; this electrical layer is distributed according to a law which can be determined in a few simple cases.

Let us consider a rectilinear cylindrical wire, the diameter of which is very small as compared with its length, and placed in its whole extent in conditions which are the same in reference to neighbouring conductors. If this wire were electrified and in equilibrium, the distribution of the superficial layer at some distance from its ends would be uniform—that is, that any portion of a surface comprised between two planes perpendicular to the axis, and at unit distance, would have the same quantity of electricity: let λ be this quantity, which may be called the linear density of the wire.

The potential V of the wire is, moreover, proportional to the total charge, and therefore to the charge of each unit length. We have then

$$V = A\lambda,$$

A being a constant which depends on the section of the wire, and on its position in reference to external conductors.

If the charge of the wire varies from one point to the other of the length, the linear density at a point, is the limit of the ratio of the charge to the corresponding lengths.

When equilibrium does not hold, it is not generally speaking exact that the potential at each point is proportional to the density; but this proportionality is evidently in particular true for cables, in which the conducting wire is surrounded by a dielectric layer of constant thickness, which in turn is surrounded by a conductor in connection with the earth. The various parts of the wire are then without appreciable action on each other, and the potential at each point is that due solely to the nearest electrical masses. If γ be the capacity of unit length of the wire—that is to say, the charge which would correspond to unit potential—the charge of a length dx at potential V would be equal to $\gamma V dx$.

220. PROPAGATION IN A WIRE WHEN THERE IS A LOSS ON THE SURFACE.—Let us still consider a cylindrical wire traversed by a current, and let us suppose that the permanent state has been attained, but that there is a loss of electricity on the surface. The flow is no longer parallel to the axis throughout the entire extent of a perpendicular section; it tends to become perpendicular to the wire close to the external surface. The equipotential surfaces S, S' (Fig. 55) are still planes throughout the greater part of their extent,

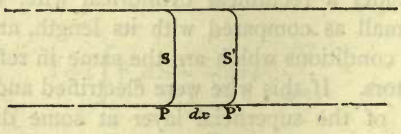

Fig. 55.

but they bend just near the edges and then join with the outer surface of the wire.

The loss which takes place at the surface is no more than a flow of electricity in the external medium; it is therefore proportional to the flow of electrical force (201). For a length dx of the wire, the charge is $\gamma V dx$ and the flow of electrostatic force is $4\pi\gamma V dx$. If c' is the coefficient of conductivity of the medium, the strength of the lateral current would therefore be $c' 4\pi\gamma V dx$. As this strength is also equal to the quotient of the potential V by the resistance of the medium, from the lateral surface in question, to the points where the potential is zero, the resistance relative to the length dx is $\frac{1}{4\pi c'\gamma dx}$ and the resistance ρ' for unit length is equal to $\frac{1}{4\pi c'\gamma}$.

The permanent state being established, the total flow of electricity through the surface S of the volume-element dx should be equal to the sum of the flows by the opposite surface S' and by the lateral surface, which gives

$$-cS\frac{dV}{dx} = -cS\left(\frac{dV}{dx} + \frac{d^2V}{dx^2}dx\right) + \frac{1}{\rho'}V dx,$$

that is to say

$$\frac{d^2V}{dx^2} = \frac{1}{cS\rho'}V.$$

The product cS represents also the reciprocal of the resistance ρ of the wire for unit length; putting $\beta^2 = \frac{1}{cS\rho'} = \frac{\rho}{\rho'}$, we get

$$\frac{d^2V}{dx^2} - \beta^2 V = 0. \tag{7}$$

This is Fourier's equation for the propagation of heat in a cylindrical bar. The integral of this equation may be put under the form

$$V = Ae^{\beta x} + Be^{-\beta x}.$$

To determine the constants A and B, we must know the potentials V_0 and V_1 at two points P_0 and P_1 at a distance of l from each other. The value of the potential at a point P, situate at a distance

x from P_0 and of $l-x$, or y, from P_1, is

$$V = V_0 \frac{e^{\beta y} - e^{-\beta y}}{e^{\beta l} - e^{-\beta l}} + V_1 \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta l} - e^{-\beta l}}. \tag{8}$$

If the point P_1 is on the ground, we have $V_1 = 0$, which gives

$$V = V_0 \frac{e^{\beta y} - e^{-\beta y}}{e^{\beta l} - e^{-\beta l}}, \tag{9}$$

If the wire is unlimited, the constant A is null, and we have

$$V = V_0 e^{-\beta x}.$$

221. The current in the wire is given by the expression

$$I = -\frac{1}{\rho} \cdot \frac{dV}{dx} = \frac{1}{\sqrt{\rho\rho'}} \frac{V_0(e^{\beta y} + e^{-\beta y}) - V_1(e^{\beta x} + e^{-\beta x})}{e^{\beta l} - e^{-\beta l}}; \tag{10}$$

if the point P_1 is connected with the earth, it becomes

$$I = \frac{V_0}{\sqrt{\rho\rho'}} \cdot \frac{e^{\beta y} + e^{-\beta y}}{e^{\beta l} - e^{-\beta l}}, \tag{11}$$

and, if the wire is unlimited,

$$I = \frac{V_0}{\sqrt{\rho\rho'}} e^{-\beta x} = I_0 e^{-\beta x},$$

I_0 being the current at the origin of the wire.

222. RESISTANCE OF A CONDUCTOR WHERE THERE IS A LOSS BY THE SIDES.—From equation (11), the total resistance offered to the electrical current from the point P_0 to the ground, allowing for the branches, is

$$R = \sqrt{\rho\rho'} \frac{e^{\beta l} - e^{-\beta l}}{e^{\beta y} + e^{-\beta y}},$$

Let us consider, as a more general case, a wire the different points in which P_1, P_2, P_3, ... are connected with the earth by conductors of resistance ρ_1, ρ_2, ρ_3, ... Let R_1 be the total resistance from the point P_1 to the ground, R_2 the resistance starting from the point P_2, ...; finally r_1, r_2, r_3, ... the resistances of the wire between the successive points of contact P_1P_2, P_2P_3, .., etc. From the point P_1 the total conductivity is equal to the sum of the conductivities which the various paths offer, which gives the equation

$$\frac{1}{R_1}=\frac{1}{\rho_1}+\frac{1}{r_1+R_2}. \tag{12}$$

We should have a series of analogous equations, and ultimately

$$\frac{1}{R_1}=\frac{1}{\rho_1}+\cfrac{1}{r_1+\cfrac{1}{\cfrac{1}{\rho_2}+\cdots\cdots+\cfrac{1}{\cfrac{1}{\rho_n}+\cfrac{1}{r_n+R_{n+1}}}}}. \tag{13}$$

This is the case of overhead telegraph wires when we take into account the leakage by the insulators. If the leakage is continuous, and R is the resistance starting from the point P, $R+dR$ the resistance from the adjacent point P′ at the distance dx, the coefficients ρ and ρ' having the same meaning as above, equation (12) becomes

$$\frac{1}{R}=\frac{1}{\cfrac{\rho'}{dx}}+\frac{1}{\rho dx+R+dR},$$

or

$$\frac{dR}{R^2-\rho\rho'}=\frac{dx}{\rho'},$$

The integral of this equation is

$$\frac{R-\sqrt{\rho\rho'}}{R+\sqrt{\rho\rho'}}=\frac{1}{C}e^{-2\beta x}. \tag{14}$$

The constant C is determined by the limiting conditions. If the resistance in the external medium starting from the end of the wire P_1 is equal to R_1, then, making $x=l$, we get

$$C=\frac{R_1+\sqrt{\rho\rho'}}{R_1-\sqrt{\rho\rho'}}e^{-2\beta l}.$$

CHAPTER II.

VARIABLE STATE.

223. APPLICATION OF FOURIER'S FORMULAS.—The problem of the propagation of electricity in a conductor when the permanent state has not been obtained—as, for instance, when a battery is discharged through a wire—offers great difficulties.

The flow of electricity which penetrates into a volume-element is not zero, since the charge varies with the time; but it cannot be asserted, *à priori*, whether the internal density varies, or, indeed, whether it still remains zero, and the increase of charge takes place only on the surface.

In the absence of adequate experimental data, the simplest idea is to pursue the analogy between the propagation of heat and that of electricity, and to try to apply Fourier's formula to the variable state of conductors. This is to assume implicitly that the flow of electricity at each point, is proportional to the electrical force at this point, or to the differential of the potential of all the acting masses. This proposition seems natural enough, if it is the case that the electrical forces really act at a distance and in an instantaneous manner, as is readily admitted in the case of universal attraction; but if, on the contrary, electrical actions are transmitted through the intervention of a medium, in virtue of what we have called the *electrical elasticity* of this medium, it is necessary to assume that the state of electrical tension (**99, 126**) is set up from layer to layer. A physical effort of this kind must necessarily require a finite time, however small this may be. This question of time, which does not affect problems of equilibrium in the permanent state, may have a preponderating influence in the phenomena of the variable state.

In other words, we may assume that the electrical force is propagated with an extremely great, but not infinite velocity, or else that the potential of an electrical mass is itself propagated with a finite velocity.

In this case it is still possible that the flow of electricity at each point is proportional to the actual electrical force, but this force will not depend solely on the position of the acting masses—it will depend also on the velocity of these masses, and the effects may be very different, according as the velocity of displacement of the acting masses is, or is not, of the same order of magnitude as the velocity of propagation of the potential.

We shall see lastly, in connection with the phenomena of electro-dynamic induction, that the displacement of electrical currents and their changes of strength, produces new electromotive forces, which can be calculated in a certain number of cases, and which may greatly modify the results relative to the variable state.

The two effects which we have mentioned are perhaps produced by the same mechanism; we shall not for the present take them into account.

With this reservation we can again apply Fourier's formulas. In any case, the results to which they lead must be so much nearer the truth the slower are the changes in the variable state; in fact, these results represent very approximately the propagation of electricity in submarine cables, and apply rigorously to Gaugain's experiments on the propagation of electricity in bodies of great resistance, such as cotton threads, or columns of oil.

224. VARIABLE STATE IN A CYLINDRICAL CONDUCTOR.—Let us consider, then, in a cylindrical conductor the volume-element dx comprised between two infinitely near sections S and S′ (Fig. 55). The potential at a point P is no longer a simple function of x—that is to say, of the position of this point—but it is also a function of the time t. During the time dt, the amount of electricity which this volume-element gains, is equal to the excess of the flow through the section S, over the flow which issues by the section S′, together with the loss by the external surface—that is to say:

$$\frac{1}{\rho}\frac{\partial^2 V}{\partial x^2}dxdt - \frac{1}{\rho'}V dxdt = \left[\frac{1}{\rho}\frac{\partial^2 V}{\partial x^2} - \frac{1}{\rho'}V\right]dxdt.$$

The increase of charge $\left(\frac{1}{\rho}\frac{\partial^2 V}{\partial x^2} - \frac{1}{\rho'}V\right)dt$ for unit length, will produce a variation of potential dV or $\frac{\partial V}{\partial t}dt$; if then we assume that the ratio of the charge to the potential, remains equal to the

capacity, as in the phenomena of statical electricity, or of the permanent state—that is to say, that the new charge is entirely on the surface,

$$\frac{1}{\rho}\frac{\partial^2 V}{\partial x^2} - \frac{1}{\rho'}V = \gamma\frac{\partial V}{\partial t}.$$

If $\alpha^2 = \gamma\rho$, the equation becomes

$$\frac{\partial^2 V}{\partial x^2} - \alpha^2\frac{\partial V}{\partial t} - \beta^2 V = 0. \tag{1}$$

If the loss by the surface may be neglected, the coefficient β^2 is zero, and we have

$$\frac{\partial^2 V}{\partial x^2} - \alpha^2\frac{\partial V}{\partial t} = 0. \tag{2}$$

We may observe, moreover, that if the loss is not zero, we may put

$$V = Ue^{-\frac{\beta^2}{\alpha^2}t} = Ue^{-\frac{t}{\gamma\rho'}};$$

equation (1) takes the form of equation (2) and becomes

$$\frac{\partial^2 U}{\partial x^2} - \alpha^2\frac{\partial U}{\partial t} = 0.$$

In both cases, the potential V is a function of x and of t, which tends to become a simple function of x when the time increases.

The general integral of this equation has been given by Fourier under several forms.

If we consider a wire of length l, originally in the neutral state, one of whose ends is in connection with the earth, while the other end is suddenly raised to potential V_0, and then kept at this electrical level, the general value of the potential at the distance x from the

origin of the wire, and at the time t from the establishment of contact, is given by the expression

$$(3)\quad \frac{V}{V_0}=\frac{e^{\beta(l-x)}-e^{-\beta(l-x)}}{e^{\beta l}-e^{-\beta l}}-2\pi e^{-\frac{t}{\gamma\rho'}}\sum_{n=1}^{n=\infty}\frac{n}{n^2\pi^2+\beta^2 l^2}e^{-\frac{n^2\pi^2}{a^2 l^2}t}\sin\frac{n\pi}{l}x,$$

which, if the loss may be neglected, becomes

$$(4)\quad \frac{V}{V_0}=\frac{l-x}{l}-2\sum_{n=1}^{n=\infty}\frac{1}{n\pi}e^{-\frac{n^2\pi^2}{a^2 l^2}t}\sin\frac{n\pi}{l}x.$$

225. Duration of the Relative Propagation.—The second equation shows that if there is no loss by the sides, the ratio of the potential at a distance x, to the potential at the origin, is the same for two different wires, at two points whose distances from the origin are proportional to the total lengths of the wires, when the ratio $\frac{t}{a^2 l^2}$ has the same value.

The time t necessary for the potential at any given point (in the middle of the wire, for instance) to attain a definite fraction of the initial potential or of the final potential, is therefore proportional to $a^2 l^2$ or $\gamma\rho l^2$—that is to say, to the square of the length of the wire, to the capacity, and to the resistance of unit length. This condition gives what may be called the *time of relative propagation* of electricity.

There is not, therefore, in the preceding conditions, a determinate velocity for the propagation of electricity as there is for sound, or for light. The apparent velocity which it has sometimes been attempted to estimate, by supposing the propagation uniform, and determining the time necessary for the electrification produced at one end of a wire to have a sensible effect at a certain distance, depends on constants characteristic of the wire, and on the sensitiveness of the means by which these electrical effects are made evident.

226. Unlimited Wire.—Fourier's general integral lends itself with difficulty to numerical applications; but we may choose simpler conditions, which really correspond to several of the observed phenomena, and enable us to find again the principal results obtained by Sir W. Thomson.

Consider an insulated wire, whose loss by the surface may be neglected, originally in the neutral state, and of an unlimited length, or at any rate of a length such that the condition at a point is not appreciably modified by that of the most distant end. At the end of the wire a constant potential V_0 is maintained. At the end of a time t, the potential at a distance x is defined by the equation

$$\frac{\partial^2 V}{\partial x^2} - \alpha^2 \frac{\partial V}{\partial t} = 0. \tag{2}$$

For a second wire placed in the same conditions as the first, and the material of which is defined by another coefficient α', we shall have in like manner

$$\frac{\partial^2 V'}{\partial x'^2} - \alpha'^2 \frac{\partial V'}{\partial t'} = 0. \tag{5}$$

Let us put $x' = mx$, $t' = nt$, m and n being constants; we may then consider the potential V′ as a function of the variables x and t, and equation (5) will become

$$\frac{\partial^2 V'}{\partial x^2} = \frac{\alpha'^2 m^2}{n} \cdot \frac{\partial V'}{\partial t}.$$

If the coefficients m and n are chosen so that

$$\alpha^2 = \frac{\alpha'^2 m^2}{n},$$

that is to say

$$\frac{\alpha^2 x^2}{t} = \frac{\alpha'^2 x'^2}{t'},$$

the potentials V and V′ satisfy the same differential equation (2), and the same limiting conditions; they represent then the same function of x and of t.

227. Hence, for unlimited wires, which is practically equivalent to wires so long that the duration of the propagation has an appreciable value, the potential V does not change when the ratio $\frac{\alpha^2 x^2}{t}$ has the same value; it is, therefore, a function of this ratio.

Hence follows the conclusion which has been established above (226), that the time required to produce a definite potential at the distance x, or more precisely, a definite fraction of the potential at the origin, is proportional to the square of the distance, and to the coefficient a^2, which is special to the wire.

In these conditions, equation (2) really only contains one independent variable, and putting

$$\frac{a^2x^2}{4t} = z^2,$$

it becomes

$$\frac{d^2V}{dz^2} + 2z\frac{dV}{dz} = 0;$$

from this we easily deduce

$$V = A\int_0^z e^{-z^2}dz + B.$$

The constants A and B are determined by the initial conditions. For $z=0$, that is to say $x=0$ or $t=\infty$, we have $V=V_0$; for $z=\infty$, that is to say $x=\infty$, or $t=0$, we have $V=0$.

We get then

$$\frac{V}{V_0} = 1 - \frac{\int_0^z e^{-z^2}dz}{\int_0^\infty e^{-z^2}dz}.$$

There is no simple expression for the integral contained in this formula, but it is met with in a great number of problems; for instance, in the theory of probabilities, and tables of it have been calculated, so that its numerical values are well known.

Between the limits 0 and ∞, for instance, it is equal to $\frac{\sqrt{\pi}}{2}$, which gives

$$\frac{V}{V_0} = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}dz. \tag{6}$$

The curve A (Fig. 56) represents the values of the ratio of the potentials $\frac{V}{V_0}$, as a function of t, by taking $a=\frac{a^2x^2}{4}=z^2t$. The ordinate remains zero for some time near the origin, and only commences to acquire an appreciable value from the period in which $t=\frac{a}{4}$. The asymptote to the curve is parallel to the t-axis at a distance from this axis equal to unity.

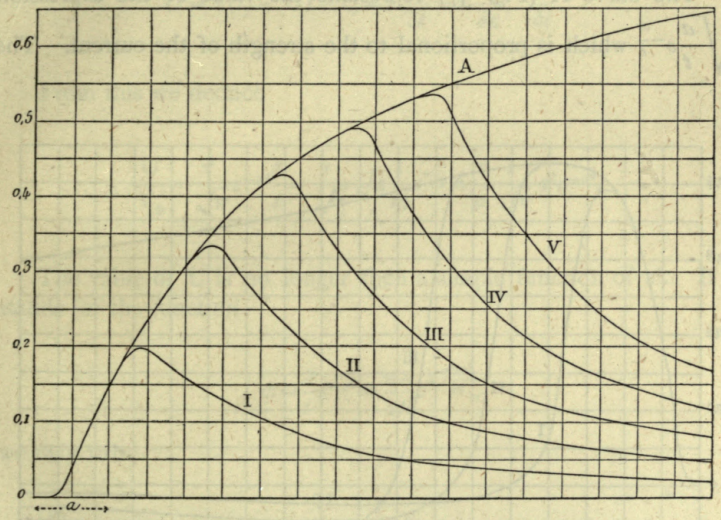

Fig. 56.

228. The expression for the current at the distance x, is

$$I=-\frac{1}{\rho}\frac{\partial V}{\partial x}=-\frac{1}{\rho}\frac{dV}{dz}\cdot\frac{\partial z}{\partial x}=\frac{V_0}{\rho}\frac{2}{\sqrt{\pi}}e^{-z^2}\frac{a}{2\sqrt{t}};$$

replacing z^2 by $\frac{a}{t}$, we get

$$(7)\qquad I=\frac{V_0 a}{\rho\sqrt{\pi a}}\sqrt{\frac{a}{t}}e^{-\frac{a}{t}}.$$

This strength is a function of the time; it is zero for $t=0$, and becomes a maximum when $\frac{\partial I}{\partial t}=0$, which gives $z^2=\frac{1}{2}$. If T is the time of the maximum we have

$$T=2a=\frac{\alpha^2x^2}{2}=\frac{\gamma\rho}{2}x^2.$$

The curve A (Fig. 57) represents the value of the expression $\sqrt{\frac{a}{t}}e^{-\frac{a}{t}}$ which is proportional to the strength of the current. The

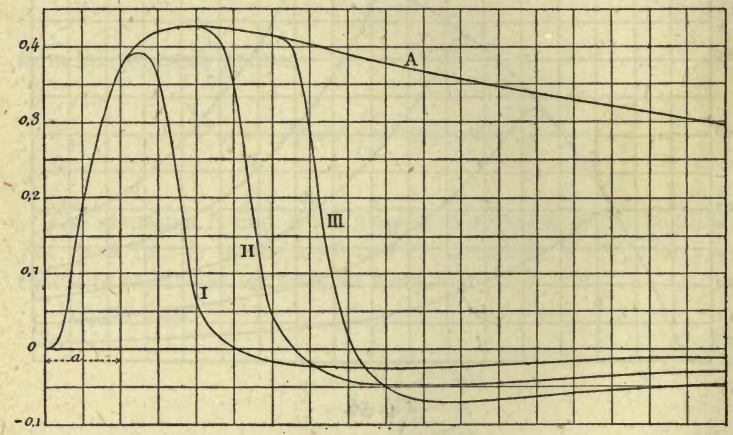

Fig. 57.

time of the maximum being proportional to α^2x^2, we see that the curve is the more bent, the greater is the distance x from the point taken as the origin.

229. MOMENTARY CONTACTS.—Let us suppose that the end of the wire is only put in momentary contact with a source at constant potential, that is to say, that this end is only raised to potential V_0 for a very brief period τ, and is then connected with the earth.

The potential at any given point will be obtained by superposing

two states, the first due to the permanent potential V_0, established at the origin of the wire at the beginning of the time, the second to the permanent potential $-V_0$, set up after the time τ. The value of the potential at the distance x, corresponding to each of those states, is the same function of the time which elapses from establishing at the origin of the wire the corresponding potential; the resultant potential U is therefore equal to $V(t) - V(t-\tau)$. If we suppose that the time τ is infinitely short, we get

$$U = V(t) - V(t-\tau) = \tau \frac{\partial V}{\partial t} = \tau \frac{dV}{dz} \cdot \frac{\partial z}{\partial t}.$$

From this we deduce

$$\frac{U}{V_0} = \frac{\tau}{\sqrt{\pi}} e^{-z^2} \frac{z}{t} = \frac{\tau}{\sqrt{\pi}} \sqrt{a} \frac{1}{t^{\frac{3}{2}}} e^{-\frac{a}{t}} = \frac{\tau}{a} \frac{1}{\sqrt{\pi}} z^3 e^{-z^2}.$$

The value of U is no longer then a simple function of z^2. If ϕ stands for the function

$$\phi = \frac{z}{t} e^{-z^2} = \frac{ax}{2t^{\frac{3}{2}}} e^{-\frac{a^2x^2}{4t}},$$

we may write

$$\frac{U}{V_0} = \frac{\tau}{\sqrt{\pi}} \phi. \tag{8}$$

230. We may, moreover, determine graphically the value of $\frac{U}{V_0}$, by taking the difference of the ordinates of the curve A, and of another identical curve which has been displaced towards the right by a quantity τ. The curves I, II, III, IV, V, represent the result of this superposition for values of τ equal respectively to a, $2a$, $3a$, $4a$, $5a$.

This construction and the formula will show that the momentary connection of the end of the wire with a source of constant potential, gives rise to a sort of *electrical wave*, which is propagated according to a somewhat complex law, and which spreads itself out as it travels.

We obtain in the same way the strengths corresponding to momentary contacts; curves I, II, III (Fig. 57), represent the law of the strengths at a point, when contacts are made for durations respectively equal to a, $2a$, $3a$.

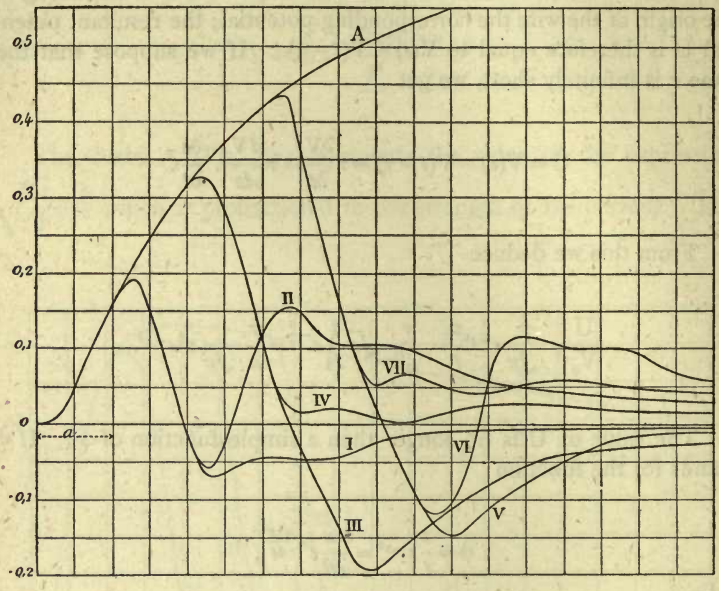

Fig. 58.

231. The period T_1, at which the maximum potential of the wave is obtained for an infinitely short contact, is determined by the condition

$$\frac{\partial U}{\partial t} = 0, \qquad \text{or} \qquad \frac{\partial \phi}{\partial t} = 0.$$

Since we have

$$\frac{\partial \phi}{\partial t} = \left(z^2 - \frac{3}{2}\right)\frac{\phi}{t},$$

the maximum takes place when $2z^2 = 3$, or

$$T_1 = \frac{2a}{3} = \frac{a^2x^2}{6} = \frac{\gamma\rho}{6}x^2 = \frac{T}{3}.$$

This time T_1 corresponds to the point of inflexion of the curve A (Fig. 56), for the condition $\frac{\partial U}{\partial t} = 0$, is equivalent to $\frac{\partial^2 V}{\partial t^2} = 0$; it is a third of the time T, required to attain maximum current at this point, with a constant potential at the end of the wire.

The time T, may therefore be considered as expressing the *duration of propagation of an electrical wave.*

232. We may, in like manner, determine either graphically or by calculation, the wave which would result when the origin of the wire is put alternately in contact with sources at potentials $+V_0$ and $-V_0$ during equal or unequal times. The curves in Fig. 58 correspond thus to alternating contacts which may be collated in the following table:

Curves.	Duration of Contacts.		
	+	−	+
I	a	a	
II	a	a	a
III	$2a$	$2a$	
IV	$2a$	a	
V	$3a$	$2a$	
VI	$3a$	a	
VII	$3a$	$2a$	a

It will be seen that, by choosing the duration of these contacts, we may obtain a far shorter wave than by a single contact. The wire is then quickly restored to the neutral state after the passage of the wave; this is the problem which it has been attempted to solve in certain cases of telegraphic communication.

The curve of contact τ may be simply constructed by adding algebraically the ordinates of the curve A, and of another curve $-A_\tau$, the origin of which has been displaced through τ. The curve corresponding to the succeeding contact τ' of the opposite sign, is obtained in like manner by the ordinates of the curves $-A_\tau$ and $+A_{\tau+\tau'}$. The curves corresponding to the contacts τ and τ' of opposite signs will therefore be obtained by the sum of the ordinates of the three curves $A - 2A_\tau + A_{\tau+\tau'}$; we thus avoid the separate construction of curves relative to the different contacts.

233. In the most general case, the potential V_0 at the origin of the wire does not pass suddenly from zero to a constant value; it is a continuous function F (θ) of the time θ, counted from the moment

the electrification begins. The element dU of the potential at the distance x and at the time t, which corresponds to the potential V_0 set up during the time $d\theta$ and to the time θ at the origin of the wire is equal (229) to $\frac{F(\theta)}{\sqrt{\pi}} \phi(t-\theta)\, d\theta$. If the total duration of the electrification is τ, we shall have for the resultant potential

$$U = \frac{1}{\sqrt{\pi}} \int_0^{\tau} F(\theta)\, \phi(t-\theta)\, d\theta.$$

It may, however, be remarked, that this expression has no meaning except for values of t, greater than τ.

If the potential at the origin of the wire varies periodically according to a simple law—if it is represented, for instance, by $V_0 \sin 2nt$, the ultimate electrical state of the wire at each point varies obviously according to the same period. The potential at the distance x may be expressed by the formula

$$V = V_0 A \sin(2nt - bx),$$

in which b is a constant, and A a function of x. Substituting this, in equation (2) we get finally

$$V = V_0 e^{-\alpha x\sqrt{n}} \sin(2nt - x\alpha\sqrt{n}). \tag{9}$$

Hence a definite phase of the potential at the origin is transmitted along the wire with a constant velocity equal to $\frac{2\sqrt{n}}{\alpha}$. In this case it may be said that there is a regular velocity of propagation, but this velocity depends on the period of electrical oscillation; the time necessary to traverse a definite length is proportional to α, and not to α^2, as has been found for the relative length of propagation considered above (225).

If the electrification at the origin, comes under a more or less complex law, and if the expression for the initial potential is decomposable into a series of simple periodic terms, the potential in the wire will be represented by a series of corresponding elementary waves; but these waves will be propagated with different velocities, and a kind of electrical dispersion will be produced analogous to the phenomenon of the dispersion of light in a refracting medium.

234. Let us suppose that the potential at the origin V_0 has alternately constant values which are equal, and of opposite signs during the very short and equal times τ, and that the operation is repeated an uneven number of times, $2n+1$ for instance—that is to say,

$$\begin{aligned} &+V_0 \text{ from } 0 \text{ to } \tau,\\ &-V_0 \text{ from } \tau \text{ to } 2\tau,\\ &+V_0 \text{ from } 2\tau \text{ to } 3\tau,\\ &\vdots \qquad\qquad \vdots\\ &+V_0 \text{ from } 2n\tau \text{ to } (2n+1)\tau. \end{aligned}$$

We have then

$$\frac{U}{V_0}=\frac{\tau}{\sqrt{\pi}}\sum_0^{2n\tau}\phi(t-\theta).$$

The different values of the function $\phi\,(t-\theta)$ are:

$$\begin{aligned} &\text{For the first contact - - - } +\phi(t-0) &&= +\phi(t),\\ &\text{For the second contact - } -\phi(t-\tau) &&= -\phi(t)+\tau\phi'(t),\\ &\text{For the third contact - - } +\phi(t-2\tau) &&= +\phi(t)-2\tau\phi'(t).\\ &\qquad\vdots \qquad\qquad\qquad\qquad \vdots &&\qquad\vdots\\ &\text{For the } (2n+1)^{th} \text{ contact } +\phi(t-2n\tau) &&= +\phi(t)-2n\tau\phi'(t). \end{aligned}$$

Adding these equations, we have

$$\sum_0^{2n\tau}\phi(t-\theta)=\phi(t)-n\tau\phi'(t)=\phi(t)\left[1-\frac{n\tau}{t}\left(z^2-\frac{3}{2}\right)\right],$$

and, therefore,

$$\frac{U}{V_0}=\frac{\tau}{\sqrt{\pi}}\phi(t)\left[1-\frac{n\tau}{t}\left(z^2-\frac{3}{2}\right)\right]=\frac{\tau}{\sqrt{\pi}}\phi(t)\left[1-\frac{n\tau}{a}z^2\left(z^2-\frac{3}{2}\right)\right].$$

The maximum value of U at the distance x is produced after a time T_n defined by the condition

$$\frac{\partial U}{\partial t}=0,$$

which gives

$$\left(\frac{2a}{3}-t\right)\left[1+\frac{3}{2}\frac{n\tau}{t}\left(\frac{2a}{3}-t\right)\right]=\frac{n\tau}{t}\left(\frac{4a}{3}-t\right).$$

If we replace t, in all the terms containing τ as a factor, by the approximate value $\frac{2a}{3}$, which corresponds to the maximum T_1 of the first wave, we get

$$\frac{2a}{3} - T_n = n\tau,$$

or finally

$$T_n = T_1 - n\tau.$$

The time at which the maximum is attained diminishes, therefore, as the number of contacts increases; the waves are thus shortened, and the duration of the phenomenon is thus materially diminished.

A series of uneven numbers of equal and alternate contacts of very short duration would thus produce along the wire an impulse in the same direction as that of a single contact, but it would be far shorter.

235. For a contact of an infinitely short duration τ, the potential at a point at the time t is

$$U = V_0 \frac{\tau}{\sqrt{\pi}} \frac{I}{t} z e^{-z^2}.$$

The value of the current is

$$I = -\frac{1}{\rho}\frac{\partial U}{\partial x} = \frac{V_0}{\rho} \cdot \frac{\tau}{\sqrt{\pi}} \cdot \frac{a}{2} \cdot \frac{I}{t^{\frac{3}{2}}} (2z^2 - 1) e^{-z^2},$$

or

$$(10) \qquad I = \frac{V_0}{\rho} \cdot \frac{\tau}{\sqrt{\pi}} \cdot \frac{a}{2a^{\frac{3}{2}}} \left(\frac{a}{t}\right)^{\frac{3}{2}} \left(2\frac{a}{t} - 1\right) e^{-\frac{a}{t}}.$$

It is easily proved that the maximum determined by the condition $\frac{\partial I}{\partial t} = 0$ is attained when

$$z^2 = \frac{a}{t} = 3 \qquad \text{or} \qquad t = \frac{a}{3}.$$

The curve I (Fig. 59) represents the values of the expression

$$\frac{1}{4}\left(\frac{a}{t}\right)^{\frac{3}{2}}\left(2\frac{a}{t}-1\right)e^{-\frac{a}{t}},$$

which is proportional to the current.

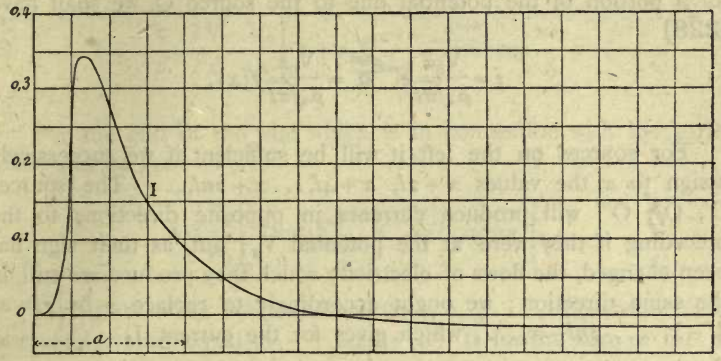

Fig. 59.

236. WIRE OF FINITE LENGTH.—To pass to the case of a wire OE (Fig. 60) of finite length l, the end of which is in connection with the earth, let us consider an unlimited wire X'X, and imagine

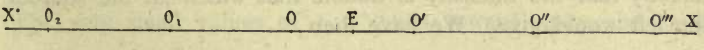

Fig. 60.

on this wire two series of sources O_1, O_2..., O', O'', O'''.. at successive distances each equal to $2l$; the first O_1, O_2,... are identical with the given sources which exists at the point O, and the others O', O'', O''',... are of the same numerical value but of opposite signs.

All these sources, O and O', O_1 and O'',... being symmetrical in pairs in reference to the point E, and of opposite signs, the potential at E will always be zero. In like manner, all the sources added, O_1 and O', O_2 and O''... are symmetrical in pairs in reference to the point O and of contrary signs: the potential

at this point will only depend on the source found there. The portion OE of the unlimited wire is therefore in the same condition as if it were alone.

The current at the point P, on the wire OE at a distance x from the origin O, is the algebraical sum of the currents which would be produced at this point if we suppose that all the sources were on an unlimited wire.

If all the sources are raised to the constant potential V_0, then, for a portion of the potential due to the source O, we shall have (228)

$$i=\frac{V_0\alpha}{\rho\sqrt{\pi t}}e^{-\frac{\alpha^2x^2}{4t}}=\frac{V_0\alpha}{\rho\sqrt{\pi t}}f(x).$$

For sources on the left it will be sufficient if we successively assign to x the values $x+2l$, $x+4l,\ldots x+2nl,\ldots$ The sources O′, O″, O‴ will produce currents in opposite directions to the preceding if they were at the potential V_0; but, as their sign has been changed, the flows of electricity which they produce are still in the same direction; we ought accordingly to replace x by $2l-x$, $4l-x,\ldots$, $2nl-x,\ldots$, which gives for the current I,

$$I=\frac{V_0\alpha}{\rho\sqrt{\pi t}}\left[..+f(4l+x)+f(2l+x)+f(x)+f(2l-x)+f(4l-x)+..\right].$$

When we have made $x=l$, that is to say, when we consider the phenomenon at the point E, at the end of the wire in connection with the earth, the expression becomes simplified, and we see directly that the intensity is double that which all the sources on the left would give. We have then

$$I=\frac{2V_0\alpha}{\rho\sqrt{\pi t}}\left[e^{-\frac{\alpha^2l^2}{4t}}+e^{-\frac{\alpha^2(3l)^2}{4t}}+e^{-\frac{\alpha^2(5l)^2}{4t}}+.....\right].$$

Putting $e^{-\frac{\alpha^2l^2}{4t}}=\nu$, we get

$$(11)\qquad I=\frac{2V_0\alpha}{\rho\sqrt{\pi t}}\left[\nu+\nu^{3^2}+\nu^{5^2}+\nu^{7^2}+....+\nu^{(2n+1)^2}+...\right].$$

The current is at first zero, since ν is zero when t is equal to zero; it then increases towards the limiting value $\frac{V_0}{\rho l}$.

The curve represented by this series would be very easily calculated, for the terms rapidly decrease when ν differs appreciably from unity.

237. Sir W. Thomson has, however, solved the problem by the help of another series, which is more easily discussed, and which follows directly from Fourier's equation (4).

According to this formula, the expression for the current at the distance x from the origin, is

$$I = -\frac{1}{\rho}\cdot\frac{\partial V}{\partial x} = \frac{V_0}{\rho l}\left[1 + 2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2 l^2}t}\cos\frac{n\pi}{l}x\right].$$

For the end of the wire which is in connection with the earth, $x = l$, and we get

$$I = \frac{V_0}{\rho l}\left[1 + 2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2 l^2}t}\cos n\pi\right].$$

Giving to n the successive values 1, 2, 3 . . . , the cosine takes alternately values equal to -1 and $+1$. If for the sake of brevity we put

$$u = e^{-\frac{\pi^2 t}{a^2 l^2}},$$

we get

$$(12)\quad I = \frac{V_0}{\rho l}\left[1 - 2(u - u^4 + u^9 - \ldots\ldots \mp u^{n2} \ldots\ldots)\right] = F(t).$$

For very small values of t, u tends toward unity, the series in the parenthesis is equal to $\frac{1}{2}$, and the current null. As the time increases, u diminishes, the series tends to zero, and the current diminishes up to a limiting value $\frac{V_0}{\rho l}$.

The series can, moreover, be easily calculated; according to Sir W. Thomson, it does not differ appreciably from its maximum value, until u is greater than $\frac{3}{4}$. If a' is the time at which this value is attained, we have

$$\frac{3}{4} = e^{-\frac{\pi^2 a'}{a^2 l^2}}, \qquad \text{or} \qquad a' = \frac{a^2 l^2}{\pi^2} l.\left(\frac{4}{3}\right).$$

We may then write

$$u = e^{-\frac{\pi^2 a'}{a^2 l^2} \cdot t} \cdot a' = \left(\frac{3}{4}\right)^{\frac{t}{a'}}.$$

The curve A (Fig. 61) represents as a function of the time and taking the final strength as unity, the curve of the current produced at that end of a wire which is in connection with the earth, when a constant potential is established at the other end.

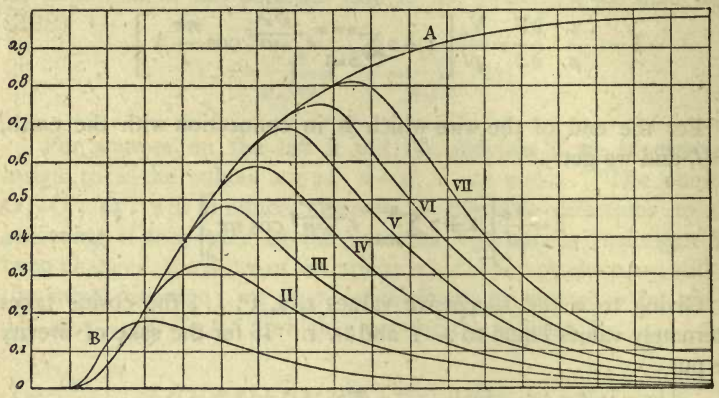

Fig. 61.

238. MOMENTARY CONTACTS.—In order to obtain the strength corresponding to the case in which the wire is connected with a source of constant potential V_0, for the time τ, it is sufficient, if, as in the case of the insulated wire, we calculate the expression

$$I_\tau = F(t) - F(t-\tau),$$

or construct geometrically the curve, the ordinate of which is equal at each point to the difference of the ordinates of the two curves F (t) and F ($t-\tau$).

Curves II, III, IV, V, VI, VII (Fig. 61) represent thus the currents arising from contacts whose durations are respectively equal to $2a'$, $3a'$, ... , $7a'$. The phenomenon appears as an electrical wave, or a momentary impulse at the end of the wire.

If the time of contact is infinitely short, the arrival curve of the current i is represented by the equation

$$i = \tau \frac{dF}{dt} = \tau \frac{dF}{du} \cdot \frac{du}{dt},$$

which gives

$$i = \tau \frac{2V_0}{\rho l} \cdot \frac{\pi^2}{a^2 l^2} (u - 4u^4 + 9u^9 - \ldots \mp n^2 u^{n^2} \pm \ldots).$$

This current is represented by the curve B (Fig. 61). It is a maximum when $\frac{di}{dt} = 0$, that is to say, when

$$u - 16u^4 + 81u^9 \ldots\ldots \mp n^4 u^{n^2} \ldots\ldots = 0,$$

an equation which gives sensibly

$$u = \left(\frac{3}{4}\right)^3, \qquad \text{or} \qquad t = 3a'.$$

239. Finally, in order to shorten the arrival waves, and to discharge the wire, the origin of the wire may be put alternately at equal potentials and opposite signs, during equal or unequal times, by connecting it with one of the poles of a battery.

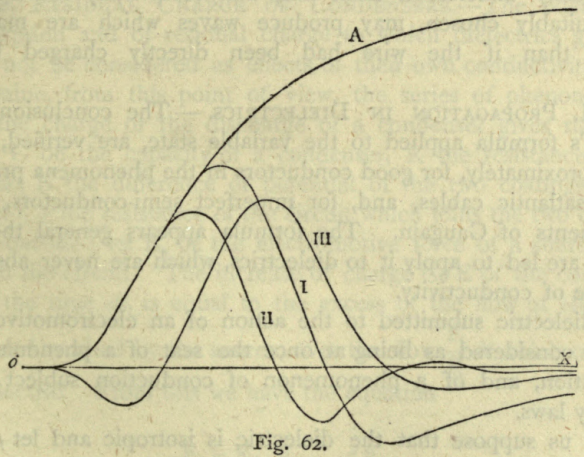

Fig. 62.

Curves I, II, and III of Fig. 62, represent the arrival waves of the alternative successive contacts :

Curves.	Duration of contact.			
	+	−	+	−
I	$4a'$	$4a'$		
II	$3a'$	$3a'$	a'	
III		a'	$4a'$	$2a'$

Without, for the present, dwelling further on this important question, it will be seen what is the nature of the problem, and what methods may be utilised for accelerating the transmission of signals in electrical wires.

240. USE OF CONDENSERS.—We may add that in practice it has been found very useful to keep the cable constantly insulated by joining each of its ends with a condenser. The battery electrifies one of the coatings of the condenser at the sending station; the other coating, which is connected with the cable, is electrified with the opposite kind, and a flow of the same kind as that which the battery would have given passes to the first coating of the condenser at the other end. The second coating of this condenser is connected with an electrometer, or is in communication with the earth by a galvanometer.

If the contact at the origin is continuous, the electrometer tends towards a maximum deviation; the galvanometer gives a deviation which increases at first and then reverts to zero, so that even for a permanent contact, the phenomenon appears as an electrical wave. It is easily understood from this, that momentary alternate contacts suitably chosen, may produce waves which are materially shorter than if the wire had been directly charged by the battery.

241. PROPAGATION IN DIELECTRICS. — The conclusions from Fourier's formula applied to the variable state, are verified, at any rate approximately, for good conductors in the phenomena presented by transatlantic cables, and, for imperfect semi-conductors, by the experiments of Gaugain. The formula appears general therefore, and we are led to apply it to dielectrics, which are never absolutely destitute of conductivity.

A dielectric submitted to the action of an electromotive force, may be considered as being at once the seat of a phenomenon of polarisation, and of a phenomenon of conduction subject to the ordinary laws.

Let us suppose that the dielectric is isotropic and let μ be its specific inductive capacity, and c its coefficient of conductivity. The general equation of induction (116) applied to a volume element dv situate at a point in which the density is ρ, gives

$$\mu\Delta V + 4\pi\rho = 0.$$

On the other hand, the variation of the charge $c\Delta dvdt$ of the element, during the time dt, produces a corresponding increase of

density, which gives the equation

$$c\Delta V = \frac{d\rho}{dt};$$

from which we deduce

$$\Delta V = -4\pi\frac{\rho}{\mu} = \frac{1}{c}\frac{d\rho}{dt},$$

and therefore

$$\rho = Ae^{-\frac{4\pi c}{\mu}t} = Ae^{-\frac{t}{T}},$$

putting $T = \frac{\mu}{4\pi c}$.

This equation shows that the density ρ constantly decreases, and that if for any reason the dielectric has received a charge in the interior, it will not retain it indefinitely; this charge will always finish by being altogether on the surface, like that of a good conductor—evidently an *à priori* conclusion.

242. RESIDUAL CHARGE OF CONDENSERS.—The phenomena of absorption and of residual charge to which dielectrics give rise should not be considered as effects of their own conductivity. Let us examine, from this point of view, the series of phenomena to which the charge or the discharge of a condenser gives rise.

Let C be the capacity of a condenser, R the resistance of the dielectric, E the difference of potential of the two coatings at the moment t, r the resistance of the circuit which joins the two coatings on the outside; let E_0 be the electromotive force of a source interposed in the circuit. The increase of charge CdE of the condenser during the time dt, is equal to the excess of the flow of electricity $\frac{E_0 - E}{r}dt$ furnished by the source, over the flow $\frac{E}{R}dt$ which traverses the dielectric. From this we have the equation

$$\frac{E_0 - E}{r} - \frac{E}{R} = C\frac{dE}{dt}, \tag{13}$$

and therefore

$$E = E_0\frac{R}{R+r}\left(1 - e^{-\frac{t}{T_1}}\right),$$

putting $T_1 = \frac{CRr}{R+r}$.

Suppose that at the moment t_1 we open the circuit, and leave the apparatus to itself during a time t_2, equation (13) reduces to

$$0 = \frac{E}{R} + C\frac{dE}{dt};$$

denoting by E_1 the difference of potential at the time t_1 between the armatures, by E_2 that which exists at the moment $t_1 + t_2$, and putting $T_2 = CR$, we have

$$E_2 = E_1 e^{-\frac{t_2}{T_2}}.$$

Suppose, lastly, that we discharge the condenser by connecting the two coatings by a conductor of small resistance ρ, we shall have the equation

$$\frac{E}{\rho} + \frac{E}{R} = -C\frac{dE}{dt},$$

and, at the end of a time t_3—that is to say, at the period $t_1 + t_2 + t_3$,

$$E_3 = E_2 e^{-\frac{t_3}{T_3}}, \qquad \text{with} \qquad T_3 = \frac{CR\rho}{R+\rho}.$$

The total loss of the condenser during the time t_3, is $C(E_2 - E_3)$; the portion which traverses the outer circuit and constitutes the discharge Q, is equal to $C(E_2 - E_3)\frac{R}{R+\rho}$, which gives finally

$$Q = E_0 \frac{CR^2}{(R+r)(R+\rho)}\left(1 - e^{-\frac{t_1}{T_1}}\right)e^{-\frac{t_2}{T_2}}\left(1 - e^{-\frac{t_3}{T_3}}\right).$$

To have a complete discharge, we must make $t_3 = \infty$; we see that we attain this complete discharge in a continuous manner, and without any of the alternatives to which condensers give rise.

Maxwell has shown that a system formed of parallel dielectric layers, and even of different dielectrical elements mixed in any way whatever, may give rise to residual charges, although each of the constituent dielectrics is destitute of this property. But the want

of homogeneity does not seem to be the sole cause of the phenomenon, and experiment shows that the existence of residual charges must in most cases be ascribed to a kind of elastic deformation which is caused by the polarization of the dielectric. It must be observed that all actions, such as repeated shocks, vibrations, sudden variations of temperature in opposite directions, which facilitate the return to the normal state, of a body which has undergone any permanent elastic deformation, also facilitate the appearance of residual charges and their return to the natural state.

The propagation of heat gives rise to no phenomenon which could be compared to the residual charge of dielectrics, and in this respect the analogy, which in so many respects is so close between the two orders of phenomena, ceases to hold.

CHAPTER III.

ENERGY OF CURRENTS.

243. Disengagement of Heat.—When a system of electrified conductors undergoes any modification whatever, without the intervention of any external force, the electrical energy in the second state is necessarily less than in the first. The energy lost during the transformation may be utilised in an equivalent form, such as a mechanical work, the raising of a weight, increase of the vis viva of the system, a change of physical state, or finally a disengagement of heat.

For any infinitely small transformation of the system in question, the loss of energy is equal to the sum of the products of each of the electrical masses, into the difference of the values of potentials at the points in which they were placed before and after the transformation.

Let us consider two points A and B kept respectively at the potentials V_1 and V_2, and on equipotential surfaces which are traversed at A and B by two corresponding portions S_1 and S_2—that is to say, cut by the same tube of flow. The quantity of electricity which traverses the two surfaces is the same; the energy lost by the current in this interval in unit time, is equal to the product of the mass of electricity which issues—that is to say, of the strength of the current I—by the difference of potentials $V_1 - V_2$, if the current goes from A to B—that is to say, by the electromotive force between these points. Hence, as a measure of the energy lost, we have

$$W = I(V_1 - V_2) = IE.$$

We shall assume as an experimental fact, that no part of this energy is employed in changing the vis viva of the electrical masses. The fact is obvious if the surfaces S_1 and S_2 are equal, for then the velocities are the same on entering and on leaving the tube. For the general case, we have already observed that the flow is parallel to the force at each point, and that therefore no effect attributable to

electrical inertia seems to intervene in the phenomena of the permanent state.

If, on the other hand, the conductor is rigid, at any rate as a mechanical whole, and if, finally, the current produces no external work, the energy is necessarily spent in the conductor itself.

244. JOULE'S LAW.—Two cases may present themselves: either the fall of potential between the points A and B is continuous, and takes place in accordance with Ohm's law; or there are, somewhere between these two points, two adjacent surfaces between which there is a sudden fall of potential, constant and independent of the strength of the current—that is to say, a constant electromotive force H. The manner in which the electrical energy is distributed along the conductor, depends on the law according to which the potential varies, and is not identical in the two cases. Wherever the variation of potential is continuous, energy is expended in a continuous manner; it is transformed into thermal energy, and gives rise to a disengagement of heat along the conductor. Wherever there is a sudden fall of potential, there is a sudden change of electrical energy, which reveals itself either by some thermal phenomenon or by some other equivalent physical effect.

Let us first consider the former case, and let us suppose that there are no variations of potential independently of the current. If R is the resistance of the conductor between two points A and B, Ohm's law gives

$$I = \frac{V_1 - V_2}{R} = \frac{E}{R}.$$

The expression for the energy expended between the two points is therefore

$$W = IE = I^2R = \frac{E^2}{R}.$$

Accordingly, *the thermal energy developed in a conductor during unit time, is equal to the product of the square of the current strength into the resistance of the conductor.* If Q be the quantity of heat, such as is measured by calorimetrical methods, and J is the mechanical equivalent of heat, we have

$$I^2R = JQ.$$

The quantity of heat disengaged is proportional to the resistance of the conductor, and to the square of the strength of the current.

This is *Joule's law.*

245. Connection between Ohm's and Joule's Laws.—This result can be arrived at in another way:

Let us consider a conductor of capacity C, a battery for instance, electrified to potential V; the value of the potential energy is (89)

$$W = \frac{1}{2}MV = \frac{1}{2}CV^2.$$

Let us now suppose this battery connected to earth by a wire whose resistance R is so great that its discharge has an appreciable duration. During the time dt, a mass of electricity dM flows out, and the potential diminishes by dV; we have

$$dM = I dt = C dV,$$

and the loss of energy in the same time is

$$dW = CV dV = V dM = IV dt.$$

Ohm's law applies if the current remains sensibly constant during the time dt; from this it follows that

$$I = \frac{V}{R},$$

and

$$\frac{dW}{dt} = IV = I^2R = \frac{V^2}{R},$$

that is to say that the energy expended in unit time is expressed by Joule's law.

We have here deduced Joule's law from the principle of the conservation of energy together with Ohm's law. Ohm's law might conversely be deduced from the same principle combined with Joule's law. For Joule's law gives

$$W = I^2R.$$

We have further

$$W = EI;$$

from which follows

$$E = IR,$$

that is to say Ohm's law.

246. We may here observe that in a multiple circuit, which does not contain localised electromotive forces, *the quantity of heat developed is a minimum, when the currents come under Ohm's law.*

Suppose, for instance, that between two points A and B, at potentials V_1 and V_2, there is a series of conducting arcs (Fig. 52). Let R be the resistance of one of them, and I the strength deduced from Ohm's law—that is to say, such that $IR = V_1 - V_2 = E$, and suppose that by a change of conditions, the strength in this conductor becomes $I + i$. The expression for the total quantity of heat developed in the new system will be

$$\sum R(I+i)^2 = \sum RI^2 + 2\sum RIi + \sum Ri^2;$$

but the product RI is a constant for each of the arcs, and on the other hand $\sum i$ is necessarily zero, if the current which terminates at the point A is not modified; the quantity of heat reduces therefore to

$$\sum RI^2 + \sum Ri^2,$$

and it is obviously a minimum, for $i = 0$—that is to say when the strength divides in the branches according to Ohm's law.

247. PELTIER'S PHENOMENON. — Let us now suppose that between two points A and B, always kept at the same potentials V_1 and V_2, the value of the potential, instead of varying in proportion to the resistances, undergoes a sudden fall $U_1 - U_2 = H$, at a point P between two adjacent surfaces, which is independent of the current; the expression for this strength will no longer be the same as in the preceding case.

If R_1 and R_2 are the resistances of the two portions AP and PB, we have thus (210)

$$I = \frac{V_1 - U_1}{R_1} = \frac{U_2 - V_2}{R_2} = \frac{V_1 - V_2 - (U_1 - U_2)}{R_1 + R_2} = \frac{E - H}{R}.$$

The total energy expended between the points A and B is

$$W = I(V_1 - V_2) = IE,$$

which gives

$$W = I(IR + H) = I^2R + IH.$$

This energy consists then of two parts; one which is proportional to the square of the strength of the current, and which heats the conductor throughout its entire length corresponding to Joule's law; and another, which is proportional to the current, is localised

at the point P. This latter is positive if the fall is in the direction of the current, and negative in the contrary case. If there is no other work than that corresponding to changes of temperature, this energy will appear as a disengagement of heat at P in the first case, and by an absorption in the second—that is, by a cooling. This is the effect which is known as *Peltier's phenomenon*, produced at the contact of the two metals.

It may be that the localised energy IH is correlated to a chemical reaction, which expends heat if H is positive, and on the other hand produces heat if H is negative, so that the changes of temperature are then merely due to the heat disengaged in accordance with Joule's law.

248. The converse of the conclusions which we have established is evident. If, at any point of the circuit, a thermal or chemical phenomenon is produced, the energy of which is proportional to the strength of the current, it may be affirmed that at this point there will be a sudden variation of potential positive or negative, according to the sign of the work, and that the variation is independent of the current. If, further, the work changes with the direction of the current, we conclude from this that the corresponding variation of potential is fixed, and is independent of the current.

Let us consider this latter case; let r be the resistance of the region in which the fall of potential is manifested, and let us suppose that only thermal phenomena are produced at that place. The quantity of heat disengaged is made up of two parts; one defined by Joule's law is expressed by I^2r, and is independent of the direction of the current; the other, due to the Peltier effect, has the value IH, and changes its sign with the direction of the current. If the current passes in one direction, the total quantity of heat disengaged is

$$IH\left(1+I\frac{r}{H}\right),$$

and if it passes in the opposite direction

$$-IH\left(1-I\frac{r}{H}\right).$$

In proportion as the current is diminished, the term $I\frac{r}{H}$ will become smaller and smaller, the Peltier effect will predominate, and the reversal of the current will more and more tend to produce equal effects and contrary signs.

A question presents itself here in reference to Peltier's phenomenon. The thermal effect observed during the passage of the current at the soldering of the two metals, measures the sudden fall of potential at this point, and it would seem as if it should measure the electromotive force of contact between them on Volta's theory. Does the result thus obtained agree with that given by other methods—the use of electrometers, for instance?

Experiment answers this question in the negative; not merely do the series of numbers obtained by the two methods disagree with each other, but the bodies themselves are not arranged in the same order; the numbers of the two series are not of the same order of magnitude; they are even sometimes of opposite signs. It is certain therefore that we are not measuring the same phenomenon in the two cases. The most plausible explanation of this discrepancy is that, in the electrostatic measurements, we are dealing with a complicated phenomenon in which the nature of the medium, necessarily interposed between the metals in contact, plays a considerable part.

249. CHEMICAL DECOMPOSITION. — Whenever a compound liquid is traversed by a current it splits up; one of the elements appears at the conductor by which the current arrives, the other at that by which it leaves. Faraday gave to this phenomenon the name *electrolysis;* the body submitted to decomposition he called an *electrolyte*, and applied the term electrodes to the two conductors by which the current enters and leaves; the former being the *positive* electrode, and the latter the *negative* electrode.*

Two conditions are necessary for the occurrence of electrolysis; the current must traverse the compound, and the compound itself must be liquid, or at any rate in the pasty state. Thus, glass at a red heat gives evident signs of decomposition, for it becomes at once a conductor, and pasty.

It is extremely remarkable that the products of decomposition only appear on the electrodes. Clausius, developing a theory which was originally propounded by Grotthüs, explains this phenomenon in a very ingenious manner. On his view the molecules of which the body is made up are in a constant state of agitation; but while the excursions of each molecule are restricted in the case of solids, these excursions may take place to any extent and in any directions in liquids. Thus the molecules of hydrogen which form part of the

* Faraday called the electrode by which the current enters the *anode*, and that by which it leaves the *cathode;* he applied the term *ions* to the elements decomposed. The *anion* is that which is liberated on the anode, the *cation* that on the *cathode;* these terms have not however, like the former, been generally adopted.

molecules of water are not invariably united to the corresponding molecules of oxygen; but, carried along in an incessant eddying, they may quit the first molecule of oxygen, to become combined with adjacent ones; and thus by a series of successive interchanges they may be carried to distances which are infinitely great in comparison with their radius of activity. In the ordinary condition, the directions of these motions are perfectly irregular; the passage of electricity imparts to them a systematic tendency, owing to which the molecules of hydrogen moving with the current are impelled towards the negative electrode; those of oxygen, on the contrary, going in the opposite direction move towards the positive electrode.

250. FARADAY'S FIRST LAW.—The first experiments on the decomposition of water by electricity appear to have been due to Troostwyk and Diemann in 1795. They employed the spark of the battery passing between two gold or platinum wires. The experiment was repeated in 1800 by Carlisle and Nicholson by means of the current of the voltaic pile. In working with sparks it is advantageous to use what are called *Wollaston's electrodes*, which consist of a platinum wire passed into a glass tube in such a way that only the mere section of the wire is in contact with the liquid. Wollaston, Faraday, Armstrong, have shown that the effect of the spark is identical with that of the battery. Whatever be the origin of the electricity, the *quantity of water decomposed is proportional to the quantity of electricity which passes.*

This law, which was enunciated by Faraday, has been more particularly verified by the electromagnetic measurement of currents; but the direct determination of the quantity of electricity by electrostatic methods also allows of a very exact demonstration. In some recent experiments Dr. Warren De La Rue discharged a condenser which had been charged to potentials 1, 2, 3, through water, and verified the exact proportionality between the quantity of electricity and the quantity of water decomposed. This proportionality enables us to regard electrolytes as measurers of electricity; the term *voltameter* is applied to an apparatus arranged so that the gases arising from the decomposition of water may be collected.

251. The work of chemical decomposition being proportional to the strength of the current, it follows, from the remark made above, that there must be somewhere in the voltameter a sudden fall of potential H, independent of the strength. The energy made available by the fall of the current at this point, is used in decomposing the water, and may be calculated in absolute value.

Let M be the quantity of electricity which has passed through

the voltameter in unit time, and P be the weight of water decomposed. These two quantities being proportional, the quotient $\frac{P}{M}=p$ expresses the weight of water decomposed by unit electricity. If a is the heat of combination of unit weight of water at constant pressure, JaP represents the energy necessary to decompose a weight of water equal to P. This energy being furnished by the fall of the current, we must have

$$W=MH=JaP;$$

from which is deduced

$$H=\frac{JaP}{M}=Jap.$$

Hence, between the two electrodes of a voltameter traversed by a current there is, besides the difference of potential due to the resistance of the intermediate conducting liquid, a sudden fall, the exact seat of which is indeterminate, and which may be produced either wholly upon one electrode, or partially on both, and which is numerically expressed by the mechanical work corresponding to the energy absorbed by that quantity of water which a unit of electricity decomposes.

252. POLARIZATION OF THE ELECTRODES.—By what mechanism is this difference of potential produced? It is clear that before the current passes, the two electrodes, if they are of the same metal, (both of platinum, for instance,) are, by Volta's law, at the same potential, which probably differs from that of water; but the sudden and opposite changes which then take place at each of the electrodes would produce in the voltameter an amount of work which is obviously zero. When the current is started, the two falls are unequal and their difference is equal to H. Following Volta's ideas, we are led to the conclusion that the surfaces in contact are modified. A deposition of the elements of the electrolyte on the electrodes gives a sufficient explanation of this modification.

For if a plate, which has been used as an electrode, or which has been immersed in a gas, is placed in water in presence of a plate of the same kind, but clean, or recently heated to redness, a difference of potential is set up between the two plates.

Let us consider, as a particular case, the decomposition of water. The first portions of gas which come in contact with the platinum, if they do not form with it a true combination, seem at any rate to be deposited there in a state of condensation in which the gas has far less potential energy than it has in the free state. This effect of condensation of the gas takes place particularly at the outset,

and then goes on progressively diminishing until the thickness of the layer becomes so great that the fresh bubbles no longer experience an action on the part of the plate, and can then escape freely.

The work of the decomposition of water only attains its normal value from this period. Hitherto the normal value has been diminished by the work of condensation in question; experiment shows that at the outset the value of this difference may be very small.

The modification which the surface of the plates thus undergoes is the cause of the phenomenon known as *polarization of the electrodes*, and which manifests itself by the development of an electromotive force opposed to that which produces the current. We can thus understand how it is that this polarization is not instantaneous, that it may increase continuously from zero to a maximum limit; and, finally, how the quantity of electricity required to produce a given state of polarization depends on the condition and dimensions of the plates. This quantity is often called the *capacity of polarization* relative to the given system.

By taking electrodes of very unequal surfaces and passing a given quantity of electricity at a given potential through the voltameter, we can produce polarization of either electrode at will; recent experiments by M. Blondlot show that the phenomenon follows the same law whatever be the direction of the current, and that for a given electrode and given electrolyte the capacity does not depend on the direction of the polarization.

253. SECONDARY CURRENTS.—When once polarization is set up, if the original current is opened and the two electrodes are themselves joined by a wire, the electromotive force of polarization H, produces a current in a direction opposite to that of the original current, but the current rapidly diminishes and finally disappears more or less completely; this current is called the *secondary current.*

It is easy to account for this phenomenon; when the two electrodes are connected by a conductor, the layer of gas gradually disappears, reforming water; the electromotive force diminishes and disappears with it; and lastly it is clear that the total quantity of electricity set in motion while the secondary current lasts, must be equal to that expended in effecting the polarization of the electrodes.

It is manifest that the current would remain constant provided the electromotive force H could be kept constant; it would be sufficient for this if the layer of gas necessary for complete polarization were maintained at the surface of the electrode. This is precisely what takes place in Grove's *gas battery.*

254. Successive Chemical Actions of the Current.—Faraday's Second Law.—Let us suppose that several Grove's cells and voltameters are arranged in series in one and the same circuit. Let n be the number of cells, n' the number of voltameters, R the total resistance of the circuit, and I the strength of the current which flows through it. In each unit of time the work done by the whole of the cells is $nJap\mathrm{I}$; that expended by the voltameters is $n'Jap\mathrm{I}$. Lastly, a quantity of work RI^2 is converted into heat in accordance with Joule's law. If there is neither positive nor negative external work, the sum of the positive works must be equal to the sum of the negative works, which gives

$$nJap\mathrm{I} = n'Jap\mathrm{I} + \mathrm{I}^2\mathrm{R},$$

from which

$$(n - n')Jap = \mathrm{IR} = (n - n')\mathrm{H}.$$

The product IR is necessarily positive; the current can only exist therefore provided that

$$n > n'.$$

The numbers n and n' are whole numbers if the polarization is a maximum in all the cells; if the polarization was incomplete in one of them, the corresponding electromotive force would only be a fraction of H, and n should then be considered as a fractional number. In all cases, the necessary and sufficient condition for the existence of the current is that n shall be greater than n'.

When the permanent state has been attained, the polarization being supposed complete in the cells as well as in the voltameters, the same work is done during the same time, positive in the one, and negative in the others. In other words, *for each unit of electricity which traverses the system, the same quantity of water is found in the couples, and is decomposed in the voltameters.*

255. Faraday's second law holds even when the polarization is not complete at all points of the circuit in question. Suppose, for instance, that in one of the couples the thickness of the layer of gas has fallen below its limiting value, and that at a given moment the electromotive force has only the value H′, which is less than H; the transport of a unit of electricity no longer represents the same work as in the others, but the relation $\mathrm{H}' = Ja'p$ is still satisfied, if by a' we represent the heat of formation of unit weight of water with

the oxygen and hydrogen in that state of partial combination in which they exist on the platinum, and the couple thus altered gives rise to the same quantity of water as all the others.

The law, moreover, is general; *the weight of elements combined or decomposed in any electrolyte is proportional to the quantity of electricity which passes;* and this whether the operation is positive or negative; whether it takes place with polarization of the electrodes, as in the decomposition of water, or of cupric sulphate with platinum electrodes; or whether the polarization can be neglected, as in the electrolysis of cupric sulphate by two copper electrodes. This statement includes as a necessary consequence that the electrolyte *never* acts as a mere conductor, and never allows any fraction of the current to pass without correlative decomposition.

In the electrolysis of cupric sulphate by two copper plates, if the two plates are really in the same condition, the electromotive force of contact of the metal with the liquid is the same on both sides, and since just as much copper is dissolved at the positive electrode as is deposited at the negative electrode, the heat produced must be equal to the heat expended. On the other hand, any difference in the state of the two plates would be shown by thermal work.

256. We may, however, state here an important restriction in the principle of the equivalence between chemical energy and electrical work. It is assumed that, at the place where the chemical action takes place, no external work, and no change of temperature is produced independently of the resistances. If this is not so, we must take into account all the physical or chemical secondary work to which the electrolysis may give rise.

In the decomposition of water, for instance, the energy of the current first brings about the separation of hydrogen and oxygen, and then does the work required by the gases in occupying a certain volume at the external pressure.

When the current arises from a Grove's battery, each element performs the same work. So long as Mariotte's law holds, the external work is always the same for the same weight of water decomposed, and therefore for the same expenditure of electricity. Within these limits the condition of equilibrium of the cells and of the electrolytes is independent of the pressure.

Mariotte's law is far from holding at very high pressures; the heat of combination of oxygen and hydrogen is thus modified, and it is known that decomposition by the battery requires the employment of far greater electromotive forces. The heat of formation of water is, moreover, a function of the temperature, and the condition of

equilibrium in a circuit may be modified, if the cells and the electrolytes are at the same temperature.

It may happen, on the other hand, that certain of the elements decomposed, experience secondary reactions which are independent of the action of the current, and give rise to an absorption or a disengagement of heat. The final result of the electrolysis would no longer be in a simple ratio with the electromotive force, and this latter could no longer be calculated from the heat of combination of the elements, taken in the condition in which they appear after the electrical operation.

257. ELECTROCHEMICAL EQUIVALENTS. — Let A, A′, A″, . . . be various electrolytes, $p, p', p'', \ldots$ the weight of each decomposed by unit electricity. These numbers are called the *electrochemical equivalents* of the various bodies, and experiment shows that they are proportional to their ordinary chemical equivalents.

If $a, a', a'', \ldots$ are the heats of combination for unit weight of each of the compounds, the elements of the combination being in the condition due to the passage of the current (that is to say, without taking into account the secondary reactions), the products $ap, a'p', a''p'', \ldots$ will be the heats of combination of the equivalents. By analogous reasoning to that in the case of water, we see that the electromotive forces relative to these various electrolytes are determined by the ratio

$$\begin{aligned} H &= Jap, \\ H' &= Ja'p', \\ H'' &= Ja''p'', \\ &\cdots\cdots\cdots, \end{aligned}$$

which give

$$\frac{H}{ap} = \frac{H'}{a'p'} = \frac{H''}{a''p''} \cdots\cdots = J.$$

It follows from this that the *electromotive force of an electrolyte is equal to the mechanical equivalent of the heat of combination of its electrochemical equivalent.*

258. E. BECQUEREL'S LAW. — The application of this law of Faraday presents no ambiguity in the case of analogous chemical compounds. If, by one and the same current, we effect the electrolysis of water, and of a series of neutral sulphates of the protoxides, for instance, the electrochemical equivalent of each metal is the weight which is deposited for the disengagement of a gramme of hydrogen; but there may be some doubt when the compounds have not the same formula. With two neutral sulphates, one of the protoxide, and the other of the sesquioxide of iron, decomposed by the

same current, it may be asked whether it is the same weight of metal, or the same weight of oxygen, which is liberated in the two electrolytes. M. E. Becquerel showed that *the metalloid determines the law*; consequently the weights of iron for the two electrolytes will be in the ratio of 3 : 2. This is also the case with the salts of other acids, the chlorides, sulphurets, etc.

259. Electrical Couples.—Let us now consider a compound circuit made up of various electrolytes, one set giving rise to positive actions and the other to negative actions. If a denotes the heat of combination for unit weight of those of the first kind, and b for those of the second, R the total resistance, and I the strength of the current, we shall have

$$\sum \mathrm{J}ap\mathrm{I} = \sum \mathrm{J}bp\mathrm{I} + \mathrm{I}^2\mathrm{R},$$

or

$$\mathrm{IR} = \mathrm{J}\left[\sum ap - \sum bp\right].$$

The product IR, which corresponds to the heat liberated in the circuit owing to the resistances, being essentially positive, the current could only exist provided that

$$\sum ap > \sum bp.$$

If this condition is not fulfilled, and all the electrolytes are at first in the natural state, the current is established the moment the circuit is closed. An incomplete decomposition polarizes the electrodes, and the current ceases as soon as the sum of the electromotive forces of polarization attains the value $\sum ap$; the system remains then in equilibrium. This is the case with a circuit formed of a Daniell's cell (263) and a voltameter; the replacement of copper by zinc in Daniell's cells gives 24·2 thermal units, while the decomposition of water requires 34·5.

260. Depolarization by Diffusion.—It may, however, happen that an extremely feeble current is then observed. This current is due to the following cause: the polarization of the electrodes of the voltameter is gradually dissipated in consequence of the diffusion of the gas; it can be seen that this diffusion will be more or less rapid according to the conditions of the experiment, but especially according to the value of the polarization itself, and its deviation in reference to the maximum polarization. The current observed in these circumstances will be that necessary to re-establish the losses due to diffusion, and to maintain the state of equilibrium which

corresponds to the maximum of polarization for the conditions of the experiment. In this way are explained the various peculiarities to which the phenomena of polarization give rise.

When we connect, with a voltameter, a source of electromotive force insufficient to produce a continuous disengagement of gas, experiment shows that the electromotive force of polarization increases with the strength of the permanent current in question, but less rapidly; that for a given value of this current, the electromotive force diminishes when the surface of the electrodes is increased; and finally, that the electromotive force is constant if the current, and the surface of the electrodes, increase in the same ratio.

261. VOLTA'S COUPLE.—A few words only are now needed to complete the theory of the battery. Volta's couple, in the strict sense of the word, consists of a plate of zinc and a plate of copper placed in water, to which a small quantity of sulphuric acid, or of any salt has been added, to make it conduct; the plate of copper being soldered to a plate of zinc which forms part of the next couple. Thus, between two terminals of the same kind there are three contacts, zinc-copper, copper-water, and water-zinc. The electromotive force may be expressed by the ordinary symbols

$$E = Zn|Cu + Cu|Aq + Aq|Zn.$$

Volta assumed that water only played the part of a conductor, and thus we shall have

$$Cu|Aq + Aq|Zn = 0,$$

and therefore

$$E = Zn|Cu.$$

On this point of view, the electromotive force of a Voltaic couple only depends on the contact zinc-copper, and these two metals joined by a layer of water are at the same potential. The alteration of the surface of the metal when in contact with the liquid or the gas, makes it very difficult to establish Volta's hypothesis in a rigorous manner.

However this may be, this alteration is so rapid, and produces such changes in the electromotive force, that the electromotive force of Volta's couple must practically be considered as depending, to a considerable extent, on the medium which forms the third element.

When the couple is closed by a conductor whose resistance is R, a current is produced the strength of which is given by the ratio

$$I = \frac{E}{R},$$

but the water is soon decomposed, oxygen goes against the current and oxidizes the zinc plate, while hydrogen goes along with the current and polarizes the copper plate; from this follows an inverse electromotive force. When the stationary condition is established, the electromotive force of polarization is E′, and the strength I′ satisfies the ratio

$$(E - E')I' = I'^2R,$$

or

$$I' = \frac{E - E'}{R}.$$

If the couple is allowed to rest, the polarization disappears slowly, owing to diffusion. When it is again closed after the lapse of some time, the current at first reappears with its original strength I (if the influence of the layer of zinc oxide may be neglected), to regain the intensity I′ after a lapse of time which is usually very short, but which may be very long if the surfaces of the electrodes are very large and the resistance of the circuit is considerable. As long as the couple is open, the difference of potential of the extremities is equal to E.

In a closed circuit the available electromotive force is E – E′. In each couple, oxide of zinc and hydrogen are produced at the expense of the zinc and of the water. As we may assume that the oxygen has passed through the gaseous state in going from the water to the zinc, it will be seen that the disposable energy of the couple, corresponds to the excess of the heat of formation of the oxide of zinc over that of the formation of water for the same weight of oxygen. If the water is acidulated, the difference corresponds to the substitution of zinc for hydrogen in sulphuric acid: this difference is about 17·7 thermal units.

The layer of hydrogen which covers the copper has also the effect of greatly increasing the resistance of the couple, which is a fresh cause for the enfeeblement of the current.

262. Unpolarizable Cells.—Mechanical means, such as the agitation of the liquid, or rubbing the copper plate with a foreign body, greatly diminish the resistance, and even the polarization, by getting rid of the greater part of the gas; the layer of gas may be completely removed by chemical action, and thus *non-polarizable* couples or cells be obtained.

A liquid which merely dissolved the hydrogen without calorific action, would increase the electromotive force by the whole amount of the work which the gas performs in filling a given volume at the external pressure; but if the hydrogen enters into a new chemical

combination, or even if we allow for the heat of solution, the electromotive force is equal to the algebraical sum of the energies produced at the two electrodes, or at the two poles of the cells.

Such, for instance, is the couple employed by Joule, in which the copper plate is covered by a layer of oxide, which the hydrogen gradually reduces. The electromotive force is equal to the difference between the heats of oxidation of the copper and of the zinc for the same weight of oxygen.

In other cases, a salt of the metal which forms the positive electrode is dissolved in the liquid; for instance, a solution of cadmium sulphate, in which is placed a plate of zinc and a plate of cadmium. The dissolved sulphate undergoes electrolysis when the circuit is closed, and a weight of cadmium is deposited on the cadmium plate which is equivalent to the zinc dissolved. The electromotive force corresponds to the heat of substitution of zinc for cadmium in the sulphate—that is, about 8·3 thermal units.

This condition lasts as long as the weight of zinc dissolved is not so great that the salt itself takes part in the electrolysis. From this time the polarization of the cell is again produced.

The electromotive force of this cell may be expressed, in terms of the electromotive forces of contact, by the following symbols:

$$E = Zn|Cd + Cd|Cd\,O.SO^3 + Cd\,O.SO^3|Zn.$$

263. Cells with Two Liquids.—In Daniell's cell two liquids are used: a concentrated solution of copper sulphate surrounding the copper plate, and a solution of zinc sulphate in which is the plate of zinc. The two liquids are separated by a membrane, such as bladder, or a vessel of porous porcelain, so as to hinder the liquids from mixing, without destroying the conductivity.

The electromotive force is

$$D = Zn|Cu + Cu|CuO,SO^3 + CuO,SO^3|ZnO,SO^3 + ZnO,SO^3|Zn.$$

While the zinc plate dissolves, the copper arising from the electrolysis of copper sulphate is deposited on the copper plate. The electromotive force corresponds to the difference between the heats of formation of the zinc sulphate and of the copper sulphate—that is to say, to the heat of substitution of the zinc for the copper in the sulphate, or 24·2 thermal units.

This cell is remarkably constant, and is one of those which undergo least change from variations of temperature.

In Grove's cell the copper is replaced by platinum: the hydrogen

is absorbed by nitric acid, and forms nitro-compounds of a lower degree of oxidation. The zinc is placed in a solution of sulphuric acid or of zinc sulphate. By substituting carbon for platinum, we get Bunsen's element.

The energy available in Grove's and Bunsen's cells represents a quantity of heat of about 47 thermal units; they have therefore almost twice as great an electromotive force as that of Daniell's cell; the liquids have, moreover, a far smaller resistance. Accordingly they are usually employed whenever very powerful currents are wanted; but the liquids change rapidly, the resistance increases, the electromotive force diminishes, and the strength of the current is soon lessened.

264. Electrostatic Phenomena in Piles or Batteries.*—The name of *pile*, frequently given to the association of several couples in connection with each other, arises from the form originally devised by Volta. Volta's pile consists of a series of double plates of zinc and copper arranged one upon the other in the same order, and separated from each other by discs of moistened cloth.

A couple consists of the whole of the bodies which exist between two zincs—that is to say, zinc, copper, water, zinc. It may be supposed that each of the zinc plates is the half of two successive couples.

If the battery commences at the bottom by a copper and ends at the top in a zinc, it will be seen that the first copper plate does not come into play. The difference of potential being equal to e for each couple, the potential will go on increasing from the bottom upwards; and if there are n couples, the electromotive force of the battery is

$$E = en.$$

265. Uninsulated Battery.—If the bottom of the battery is connected with the earth by conductors whose influence may be neglected, the top disc A has a potential $V_a = E = ne$ which is proportional to the number of couples. This is easily verified, either by means of an electrometer, or by measuring charges given to a condenser.

266. Insulated Battery.—If the battery, which we will suppose is formed of identical and equidistant couples, has not been connected with the ground, or at any rate after such a length of time that it has attained equilibrium, its total charge will be zero, and the distribution of potentials will be symmetrical in reference to the middle. We

* The term *battery* is more generally used in this country and will be here adopted.—Trans.

shall have therefore for the ends A and B, or the two poles,

$$V_a + V_b = 0,$$
$$V_a - V_b = E,$$

and accordingly

$$V_a = -V_b = \frac{E}{2}.$$

Suppose that we give an extra charge M′ to the battery; this charge will distribute itself as it would on an ordinary conductor of the same shape, and will produce a constant potential V′ in the interior, such that if P is the capacity of the battery,

$$V' = \frac{M'}{P}.$$

The potential V′ being added everywhere to the original potential will not affect the law of contacts. Hence, at the top A we shall have

$$V_a = \frac{E}{2} + V' = \frac{ne}{2} + V',$$

and, on the mth couple from the bottom,

$$V_m = \left(m - \frac{n}{2}\right)e + V'.$$

Let us now suppose that we connect the mth couple of an insulated battery, whose total charge is zero, with a conductor whose capacity is C. This will take a charge M; there will be a fall of potential V in every point of the battery, so that if V_m is the new potential of the couple in question, we shall have

$$M = CV_m = PV.$$

from which we get

$$V_a = \frac{E}{2} - V = \frac{E}{2} - \frac{C}{P}V_m = \frac{ne}{2} - \frac{C}{P}V_m.$$

We have, moreover,

$$V_a - V_m = (n - m)\,e.$$

Eliminating the intermediate potential V_m from these two equations, we get

$$V_a = e \frac{\frac{P}{C} \cdot \frac{n}{2} + n - m}{1 + \frac{P}{C}}.$$

If the mth couple is connected with the earth, we clearly have

$$V_a = e(n - m).$$

The distribution of potentials on any given battery, symmetrical or not, would be determined in the same way. In the latter case, the neutral point of the insulated and uncharged battery is no longer in the middle.

267. REPRESENTATION OF POTENTIALS IN THE INTERIOR OF THE BATTERY.—Let us represent the battery by a straight line such that each portion of the length is proportional to the resistance of the part which it represents, and at each point draw an ordinate proportional to the potential at this point. Let us suppose the case to be that of a battery of Volta's couples, the potential increases by a constant quantity at each zinc-copper contact; the curve will show then at the corresponding points a sudden change of the ordinate, which is always the same. If the battery is open, the potential is constant in the battery from one contact to the following; the curve representing the potentials will be formed of a series of equidistant steps like those of a ladder. The line of zero potential passes through the middle, if the battery is insulated, or it passes through any given point which is connected with a conductor of some capacity or with the ground.

Three cases may present themselves in the case of a closed battery:

1. The interpolar conductor has a resistance which may be neglected in comparison with that of the battery. The two poles are sensibly at the same potential, and each contact produces the same variation of potential; but from one contact to the following there is a progressive fall of precisely equal amount. If n is the number of couples, and r the resistance of each of them, Ohm's law gives

$$I = \frac{ne}{nr} = \frac{e}{r};$$

the current is the same as with a single couple.

2. The resistance of the battery may be neglected in comparison with that of the interpolar conductor. The variation of potentials in the interior of the battery is almost exactly the same as if it were open. On the outside the fall of potential is continuous; and if R is the resistance of the interpolar conductor, the current is

$$I = \frac{E}{R} = n\frac{e}{R};$$

it is then proportional to the number of couples.

3. Finally, if the resistance of the interpolar is of the same order as that of the battery, the potential rises by a constant quantity at each contact, and sinks continuously, but to a less extent, from one contact to the next; the difference of potentials has a finite value, less than in the case of an insulated battery, but which is greater the greater the resistance of the interpolar, and by Ohm's law, the current is

$$I = \frac{E}{R + nr}.$$

268. BATTERY PLACED IN A CONDUCTING MEDIUM.—We have hitherto supposed that there is no loss of electricity by the lateral surface of the battery. Imagine that a battery, of Volta's original construction, made up with infinitely thin plates, is placed in a conducting medium, and that electricity flows both from the sides and from the ends; this would be the case of a battery immersed in water, if the effects of polarization are neglected. Let ϵ be the electromotive force of the battery for unit length, ρ the internal and ρ' the external resistance for unit length (220).

The flow of electricity is still parallel to the generating surfaces for the greater extent of each normal section of the battery, and part escapes at each point, so that the equipotential surfaces are plane and agree with the lateral surface, as in Fig. 55.

Between the infinitely near points P and P', whose potentials are V and V', the strength I of the current in the interior satisfies the equation

$$V' = V + \epsilon dx - I\rho dx.$$

The current at each point is therefore given by the equation

$$I = \frac{1}{\rho}\left(\epsilon - \frac{dV}{dx}\right). \tag{1}$$

Let us assume that the permanent state has been attained, and consider two successive layers. The flow of electricity which traverses the first is equal to the sum of the flow I′ which traverses the second, and of the flow i which escapes by the lateral surface, that is

$$I = I' + i,$$

or

$$i = I - I' = -dI.$$

As we have

$$i = \frac{1}{\frac{\rho'}{dx}} V = \frac{V\,dx}{\rho'},$$

it follows from equation (1) that

$$\frac{1}{\rho}\frac{d^2V}{dx^2} = \frac{1}{\rho'}V.$$

Making $\beta^2 = \frac{\rho}{\rho'}$, this equation becomes

$$\frac{d^2V}{dx^2} - \beta^2 V = 0,$$

the same as for the permanent state of a wire when there is an escape at the surface (220). To determine the constants of the integral

$$V = Ae^{\beta x} + Be^{-\beta x},$$

let us suppose that the lengths are calculated from the middle of the battery, and that, the whole being symmetrical, the potential is zero when $x = 0$; it follows that

$$V = A\,(e^{\beta x} - e^{-\beta x}).$$

If $\pm V_1$ is the potential at the ends of the battery and l its length, we have

$$(2) \qquad V = V_1 \frac{e^{\beta x} - e^{-\beta x}}{e^{\frac{\beta l}{2}} - e^{-\frac{\beta l}{2}}}.$$

269. The expressions for the current in the battery, and that for the lateral current i, are

$$(3)\qquad I=\frac{1}{\rho}\left(\epsilon-\frac{dV}{dx}\right)=\frac{1}{\rho}\left[\epsilon-\frac{V_1\beta}{e^{\frac{\beta l}{2}}-e^{-\frac{\beta l}{2}}}(e^{\beta x}+e^{-\beta x})\right],$$

$$i=\frac{V}{\rho'}dx=\frac{1}{\rho'}\frac{V_1}{e^{\frac{\beta l}{2}}-e^{-\frac{\beta l}{2}}}(e^{\beta x}-e^{-\beta x})dx.$$

In order to determine the potential $\pm V_1$ of the ends, we must estimate the current which flows through each of them.

We have then, if R_1 is the resistance of the medium measured from the ends,

$$I_1=\frac{V_1}{R_1}=\frac{1}{\rho}\left[\epsilon-\frac{V_1\beta}{e^{\frac{\beta l}{2}}-e^{-\frac{\beta l}{2}}}\left(e^{\frac{\beta l}{2}}+e^{-\frac{\beta l}{2}}\right)\right],$$

which gives

$$(4)\qquad \frac{V_1}{e^{\frac{\beta l}{2}}-e^{-\frac{\beta l}{2}}}=\frac{\epsilon}{\left(\frac{\rho}{R_1}+\beta\right)e^{\frac{\beta l}{2}}-\left(\frac{\rho}{R_1}-\beta\right)e^{-\frac{\beta l}{2}}},$$

$$(5)\qquad I=\frac{\epsilon}{\rho}\left\{1-\frac{\beta}{\left(\frac{R_1}{\rho}+\beta\right)e^{\frac{\beta l}{2}}-\left(\frac{R_1}{\rho}-\beta\right)e^{-\frac{\beta l}{2}}}(e^{\beta x}+e^{-\beta x})\right\}.$$

If the external medium is an insulator, $\rho'=\infty$ and $\beta=0$. The second term of the current appears then in an indeterminate form, but we get finally the ordinary expression,

$$I=\frac{l\epsilon}{\rho l+2R_1}=\frac{E}{\rho l+2R_1}.$$

The total resistance of the battery, and of the medium,

measuring from the point P, is given (222) by the expression

$$\frac{R - \sqrt{\rho\rho'}}{R + \sqrt{\rho\rho'}} = \frac{1}{C_1} e^{-2\beta x}.$$

We can determine the constant C_1 by the condition that this resistance becomes equal to R_1, for $x = \frac{l}{2}$, which gives

$$C_1 = \frac{R_1 + \sqrt{\rho\rho'}}{R_1 - \sqrt{\rho\rho'}} e^{-\beta l}.$$

270. ELECTROCAPILLARY PHENOMENA.—The preceding experiments have shown that any modification of the surface of contact of two bodies brings with it a variation in the electromotive force. This may be considered as a general law, and we must assume *a priori* that there is a relation between the electromotive forces of contact of two bodies, and any other property dependent on the state of the surfaces.

If, for instance, we use a surface of mercury as negative electrode to decompose water, the mercury becomes polarized—that is to say that the difference of potential at the contact of the two liquids increases with the external electromotive force until the disengagement of bubbles of gas begins. The capillary properties of mercury (that is to say its surface tension), depend simply on the state of the surface, and must therefore change with the polarization.

M. Lippman's experiments have shown that this is the case. The capillary tension of mercury in contact with acidulated water, increases at first with the electromotive force of polarization until this reaches 0.9 of the electromotive force of a Daniell's cell, and then diminishes in proportion as the polarization increases.

Reasoning and experiment alike show that the converse of this is true. If by any mechanical process whatever, the surface of the mercury is deformed, and therefore the surface tension of contact of the two liquids is made to vary, the difference of potential changes at the same time; during the deformation the potential varies in such a way that the surface tension which corresponds to it tends to oppose the motion produced.

CHAPTER IV.

THERMOELECTRIC CURRENTS.

271. SEEBECK'S DISCOVERY.—We have seen that a closed circuit consisting of several metals at the same temperature, cannot give rise to a current; but this law no longer holds if the different parts of the circuit, and particularly the solderings of the metals, are not at the same temperature. The circuit is then traversed by what is called a *thermoelectrical current.*

This important discovery was made by Seebeck in 1821.

In a circuit formed of a bar of bismuth, the ends of which are joined by a strip of copper, the current goes from the bismuth to the copper through the heated soldering; the copper is then said to be negative to the bismuth. With a couple antimony-copper, the current is reversed—it goes from copper to antimony through the heated junction; the antimony is accordingly negative in reference to copper.

It is natural to suppose that the metals could be classed in a regular series based on this new property, and that antimony, which is negative to copper, is much more negative to bismuth. This, in fact, is what experiment shows, and the electromotive force for the same temperature at the junctions, is greater with the couple bismuth-antimony, than with either of the two couples bismuth-copper or copper-antimony.

The electromotive force of a thermoelectrical couple may be obtained by breaking the circuit at a point outside the junctions, and determining the difference of potential at the two ends.

In a circuit consisting of a single homogeneous metal, it is impossible to set up an electrical current by variations of temperature, whatever may be the shape and section of the conductors near the heated points. Currents might, however, be produced if the

metal has either a temporary or permanent dissymmetry in its physical properties, on either side of the heated part.

272. LAWS OF THERMOELECTRICAL CURRENTS.—Without discussing the experiments which demonstrate these special points, and which have served to establish the laws of the phenomenon, we shall confine ourselves to giving the laws themselves.

I. LAW OF VOLTA.—*There is never a current in any metallic circuit all of whose points are at the same temperature.*

For the algebraical sum of all the electromotive forces of contact is necessarily zero since the metals obey the law of successive contacts (189).

II. LAW OF MAGNUS.—*In any homogeneous circuit there is never a permanent current, whatever may be the shape of the conductor, and whatever the variations of temperature which exist between the different points of the circuit.*

This law leads to the conclusion, either that the variation of temperature from one point to another determines no difference of potential between these two points, or that this difference, if it exists, only depends on the temperatures themselves, and not at all on the law of variation.

From the hottest part of the circuit to the coldest, we find, in fact, by two different paths, the same fall of temperature, but with variations entirely independent on either side. If there are variations of potential in the circuit, the sum of these variations is null; hence between the two temperatures t and t', the total variation of the potential must be the same on each side.

It follows from the law of Magnus *that the electromotive force only depends on the temperature of the two junctions, and not at all on the distribution of temperatures in the conductors which separate them.*

We shall represent by $E_t^{t'}(AB)$ the electromotive force of the two metals A and B when the junctions are at the temperatures t and t', the current going from A to B across the hottest junction at the temperature t'. This electromotive force is a function of the two temperatures t and t'.

III. LAW OF SUCCESSIVE TEMPERATURES (BECQUEREL).—*For a given couple the electromotive force corresponding to any two temperatures t and t' of the two junctions, is equal to the sum of the electromotive forces, which correspond to the temperatures t and θ on the one hand, and θ and t' on the other, θ being a temperature between the two former.*

This law may be expressed as follows:

$$E_t^{t'} = E_t^{\theta} + E_{\theta}^{t'}.$$

We have already learnt that the electromotive force only depends on the temperature of the two junctions; this latter law shows that the electromotive force may be expressed by the difference of two terms, one of which only contains the temperature t and the other t', these two terms being the values of the same function of the temperature.

We may then write

$$E_t^{t'} = F(t') - F(t).$$

IV. Law of Intermediate Metals (Becquerel).—*If two metals* A *and* B *are separated in a circuit by one or more intermediate metals, with all intermediate junctions kept at the same temperature* t, *the electromotive force is the same as if the metals were directly connected, and the junction raised to the same temperature* t.

The law of intermediate metals may be expressed by the equation

$$E_t^{t'}(AB) = E_t^{t'}(AC) + E_t^{t'}(CB).$$

For if two metals A and B are connected at the hot junction by an intermediate metal C, from the law of Magnus we may suppose that a point P of this third metal is at the lower temperature t, and interpose, in like manner, at the cold junction, a piece of the metal C kept at the temperature of this junction. We have then the two couples AC and CB in the circuit between the same limits of temperature; the electromotive force is that which would be directly produced between the metals A and B.

This law is of great practical importance; it shows that the soldering at the junction of two metals has no influence on the phenomena to which they give rise.

V. Phenomena of Inversion.—In the case of some thermo-electric couples, the strength of the current increases continuously as the temperature of the heated junction is raised, that of the cold junction remaining unchanged. The couple is said to *work uniformly* when the electromotive force is proportional to the difference of the temperatures of the two junctions.

In most cases, on the contrary, the electromotive force of the couple, after having passed through a maximum, becomes null, and then changes its sign.

Hence, at a certain temperature, there is an *inversion* of the current, and the strength then increases continuously without showing a fresh inflection. This phenomenon was discovered by Cumming in 1823.

Gaugain found that the temperature of inversion depends on that of the cold junction, and that *for a given couple the mean of the temperatures of the two junctions at the moment of inversion is constant and always equal to the temperature of the maximum strength.*

273. GRAPHICAL REPRESENTATION OF THE PHENOMENA. — Gaugain, in a remarkable research on thermoelectrical phenomena, represents their course by a graphical method by which the preceding laws may be readily verified. Taking for the abscissa the difference $t - t_0$ of the temperatures of the two junctions (the cold one having a constant temperature of 20°), he erects at each point an ordinate proportional to the corresponding electromotive force.

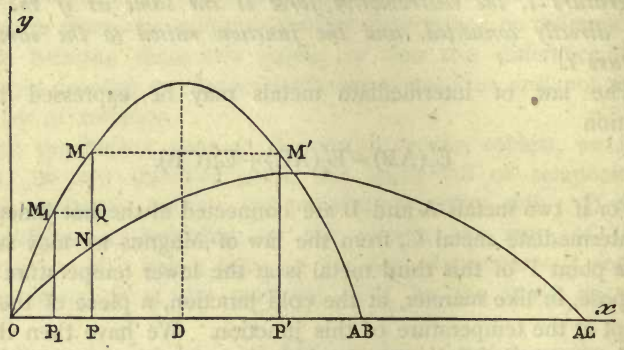

Fig. 63.

The following properties are observed in these curves (Fig. 63):

1. They are symmetrical in reference to the maximum ordinate, which verifies the law relative to the temperature of inversion; for if t_m is the temperature of the maximum, and t_i that of inversion,

$$t_m = \frac{t_0 + t_i}{2}.$$

These curves are calculated by Gaugain to be branches of hyperbolas with a vertical axis, but they may be replaced by parabolas; the difference of the ordinates calculated for the two curves are of the same order as experimental errors; both represent equally

well the results of experiment. Theory indicates, as we shall see later, that the curve which represents electromotive forces as a function of temperature must, in effect, be a parabola.

2. If a horizontal line is drawn through a point M_1, which corresponds to the temperature t_1, the ordinates, counted from this straight line, will represent electromotive forces relative to the temperature t_1 for the cold junction. The law of successive temperatures is thus found to be verified, for we have

$$MP = MQ + QP = MQ + M_1P_1,$$

that is to say,

$$E^{t}_{t_0} = E^{t}_{t_1} + E^{t_1}_{t_0}.$$

The temperature of inversion corresponds to the point where the new line of the abscissa meets the curve. If OP represents the temperature of the cold junction, OP′ will be that of inversion; it will be seen that it depends on the temperature of the cold junction.

3. If the curves AB and AC represent electromotive forces for couples formed of a metal A associated respectively with two metals B and C, the difference MN of the ordinates of the two curves represents the electromotive force of the couple formed by the two metals B and C. The relation MP = PN + NM is therefore equivalent to the equation

$$E(AB) = E(AC) + E(CB),$$

which expresses the law of intermediate metals.

274. CONCLUSIONS FROM VOLTA'S LAW. — Disregarding the principle of inversion, we may look upon the preceding laws as consequences of the principle of Volta—that is to say, that *there is an electromotive force at the contact of two metals, and that this electromotive force is a function of the temperature.*

On this view, the electromotive force of a couple is the algebraical sum of the two electromotive forces in contrary directions which exist at the two junctions.

Let us agree to represent by the symbol $\frac{A|B}{t}$ the electromotive force H of contact of two metals A and B, at the temperature t, we shall have

$$E^{t_2}_{t_1}(AB) = \frac{B|A}{t_2} - \frac{B|A}{t_1} = H_2 - H_1.$$

Let I be the current which traverses the circuit whose total resistance is R. In unit time, the work withdrawn from the heated junction is IH_2, and the work expended at the cold junction is IH_1; the difference of these two works is transformed into thermal energy, which is disengaged in the circuit in accordance with Joule's law, and we have

$$IH_2 = IH_1 + I^2R,$$

whence

$$I = \frac{H_2 - H_1}{R}.$$

The system may therefore be looked upon as a heat engine, the boiler of which yields a quantity of heat Q_2 given by the equation $JQ_2 = IH_2$, while the condenser absorbs a smaller quantity of heat Q_1, defined in like manner by the equation $JQ_1 = IH_1$, the difference of these two quantities being employed to heat the circuit, from which it follows that

$$J(Q_2 - Q_1) = I^2R.$$

The law of Magnus is contained in the hypothesis that there is no electromotive force at the junctions.

The law of successive temperatures follows from the identity

$$E_{t_1}^{t_2}(AB) = \frac{B|A}{t_2} - \frac{B|A}{\theta} + \frac{B|A}{\theta} - \frac{B|A}{t_1} = E_{\theta}^{t_2} + E_{t_1}^{\theta}.$$

Lastly, the law of intermediate metals is also evident, for, by definition, we have

$$E_{t_1}^{t_2}(AC) + E_{t_1}^{t_2}(CB) = \frac{C|A}{t_2} - \frac{C|A}{t_1} + \frac{B|C}{t_2} - \frac{B|C}{t_1}.$$

On the other hand, Volta's law of tensions gives, for any given temperature,

$$\frac{B|C}{t} + \frac{C|A}{t} = \frac{B|A}{t};$$

the preceding equation thus becomes

$$E_{t_1}^{t_2}(AC) + E_{t_1}^{t_2}(CB) = \frac{B|A}{t_2} - \frac{B|A}{t_1} = E_{t_1}^{t_2}(AB).$$

275. Consequences of Inversion.—The principle of Volta, restricted to the contact of bodies of different kinds, is not sufficient to explain the phenomena of inversion.

Let us consider, in fact, a circuit consisting of two metals A and B. In order to account for inversion as a mere effect of contact, we must assume that the difference of potential of the junction at first increases with the temperature, passes through a maximum, then diminishes, and, at the temperature of inversion, becomes equal to the difference of potential at the cold junction. The value of H_2 would then continue to decrease; and next, the current having changed its sign, the play of the electrical forces would produce a disengagement of heat at the hot junction, and an absorption at the cold one, besides the heating of the circuit in virtue of Joule's law. We may imagine that the causes of the cooling of the circuit are so diminished that it is possible to dispense with the source of heat, and that the mere passage of the current would be sufficient not merely to keep up the temperature of the hot junction, but even to increase it, and to diminish that of the cold one, the effect of which would be to intensify the current. In this way we should have realised a metallic circuit possessing the remarkable property of transferring heat from the colder to the hotter parts without any expenditure of energy. And although such a result is not so obviously impossible as that of the impossibility of perpetual motion, it is incompatible with the general course of thermal phenomena; it is, moreover, in direct contradiction with Carnot's principle.

If, on the other hand, thermoelectrical currents were merely due to the electromotive forces at the junctions, Carnot's principle would necessitate that all couples had a uniform course.

Let us imagine, for example, that a thermoelectrical couple working between the temperatures t_1 and t_2 is connected with an electrolyte whose electromotive force of decomposition is E; we shall have the ratio

$$IH_2 - IH_1 = I^2R + IE,$$

or

$$J(Q_2 - Q_1) = I^2R + IE.$$

If the current I is very small, and the resistance R moderate, the term I^2R may be neglected, the opposing electromotive force E is very little less than $H_2 - H_1$, and the excess of heat furnished by the hot source is employed in performing the external work IE. Let us suppose that, by any means, E is made to increase to the

value E', which is little more than $H_2 - H_1$, the direction of the current would change; if the absolute value of the strength remains the same, the same quantities of heat would be put in play at each junction, but in opposite directions, and the electrolyte would produce heat instead of absorbing it. In the case of a very feeble current, the thermoelectrical pile would behave as a reversible calorific engine, and we may apply the principle of Carnot. If T_1 and T_2 are the absolute temperatures of the two junctions, the quantities of heat Q_1 and Q_2, absorbed or furnished by the two sources according to the working of the machine, must be proportional to the absolute temperatures T_1 and T_2, and we should have

$$\frac{Q_2}{T_2} = \frac{Q_1}{T_1},$$

or, A being a constant,

$$\frac{H_2}{T_2} = \frac{H_1}{T_1} = A.$$

From this would follow

$$\frac{H_2 - H_1}{T_2 - T_1} = \frac{E^{t_2}_{t_1}}{T_2 - T_1} = A,$$

and therefore

$$E^{t_2}_{t_1} = A(T_2 - T_1).$$

Hence the electromotive force of all couples should be proportional to the difference in temperature of the two junctions; all couples would have a uniform course, and the phenomena of inversion could never be met with.

276. Sir W. Thomson's Theory.—Volta's principle is therefore incapable of giving a complete explanation of thermoelectrical phenomena; we must accordingly assume the existence of electromotive forces other than those of contact, and capable, like them, of producing reversible thermal phenomena.

The least changes in the physical condition of metals, such as tempering, torsion, or traction, etc., modify their electrical properties; it is accordingly natural to assume that the contact of two parts of the same metal at different temperatures also gives rise to a difference of potential.

The electromotive force resulting from variations of temperature is null in a homogeneous wire (law of Magnus), for the total fall of potential on either side of the maximum is of the same value; but this compensation no longer holds on each side of the junction of two different metals, and we must take into account the continual change of potential which variations of temperature determine along conductors.

To give greater definiteness to these conceptions, let us consider a copper-iron pair, for example, working between the temperatures t_1 and t_2, and let H_1 and H_2 (Fig. 64) be the electromotive forces of contact at these two temperatures; suppose, further, that the potential has increased along the copper C_u, in consequence of a rise of temperature from t_1 to t_2, by a quantity c independent of the strength of the current; and that conversely there is a fall of potential

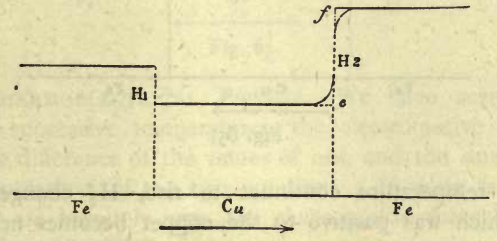

Fig. 64.

f, on the iron F_e for the same excess of temperature; the potential near the hot junction will be higher by a quantity $f+c=h$, and the electromotive force of the couple will now be

$$E = H_2 + h - H_1.$$

We have implicitly assumed that the temperature t_2 is lower than the temperature of inversion. The current goes from copper to iron through the hot junction; the thermal energy absorbed at the hot junction, as well as on the adjacent points, is equal to $(H_2+h)I$, and that which is expended at the cold junction H_1I.

The lower temperature t_1 being fixed, the electromotive force of the couple will increase as long as H_2+h increases—that is, so long as

$$\frac{dH_2}{dt} + \frac{dh}{dt} > 0,$$

and the maximum will take place at the temperature t_m, which is evidently independent of t_1, and is defined by the condition

$$\frac{dH_2}{dt}+\frac{dh}{dt}=0,$$

We shall see that at this instant the value of H_2 is zero, and that it then becomes negative. The difference of potential near the junction is then simply due to the variations of temperature on the two metals (Fig. 65).

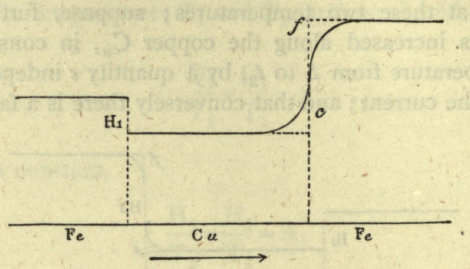

Fig. 65.

As the temperature continues to rise, H_2 changes its sign; the iron which was positive to the copper becomes negative; the

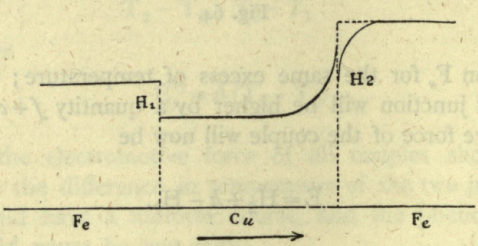

Fig. 66.

distribution of potential is represented by Fig. 66, and heat is disengaged at the two junctions.

Inversion takes place at the moment at which

$$H_2+h=H_1.$$

The electromotive force changes its sign at a higher temperature at the heated junction, and we have

$$H_2+h<H_1.$$

In this case, which is represented by Fig. 67, the current absorbs thermal energy at the two junctions, IH_2 at the hot one, IH_1 at the cold one, and a quantity Ih is liberated at those points where the temperature varies.

Such is a general idea of Sir W. Thomson's theory, the mathematical consequences of which we shall proceed to develop. We shall apply the term *Thomson effect* to the difference of potential due to the differences of temperature which form the basis of this theory.

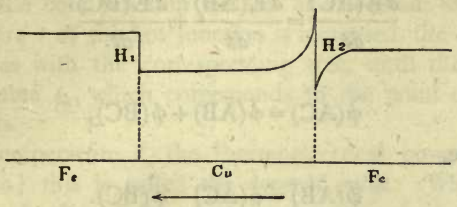

Fig. 67.

277. THERMOELECTRICAL POWERS.—We have seen that, by the law of successive temperatures, the electromotive force of a couple is the difference of the values of one and the same function for the temperatures of the two junctions. If these temperatures t and $t+dt$ are infinitely near, the electromotive force is infinitely small, and is expressed by

$$dE = F(t+dt) - F(t) = \frac{dF}{dt}\,dt;$$

we may therefore write

$$\frac{dE}{dt} = \phi(t). \tag{1}$$

Sir W. Thomson calls the function $\phi(t)$ *the thermoelectrical power of the two metals at the temperature t.* This function is nothing but the angular coefficient of the tangent to Gaugain's curves. We can deduce from it the electromotive force of the couple for the temperatures t_1 and t_2 of the two junctions by the formula

$$E_{t_1}^{t_2} = \int_{t_1}^{t_2} \phi(t)\,dt.$$

278. This function possesses a remarkable property, in virtue of which thermoelectrical phenomena may be very simply expressed.

The thermoelectrical power of two metals A *and* B *at a temperature t is equal to the difference of the thermoelectrical powers of the same metals* A *and* B *in reference to any third metal* C.

For the law of intermediate metals gives the equation

$$E(AC) = E(AB) + E(BC).$$

From which we deduce

$$\frac{dE(AC)}{dt} = \frac{dE(AB)}{dt} + \frac{dE(BC)}{dt},$$

or

$$\phi(AC) = \phi(AB) + \phi(BC),$$

and, therefore,

$$\phi(AB) = \phi(AC) - \phi(BC). \qquad (2)$$

If, then, we know the thermoelectrical power of different metals in reference to a standard metal X, it will be easy to deduce from this the thermoelectrical power of any two metals by the formula

$$\phi(AB) = \phi(AX) - \phi(BX).$$

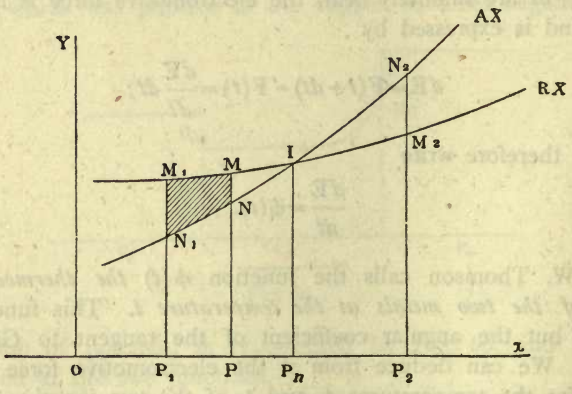

Fig. 68.

Let AX (Fig. 68) be the curve which represents the value of ϕ as a function of t for the two metals A and X, and BX the analogous curve for the two metals B and X; from equation (2) we shall have

$$\phi(AB) = MP - NP = MN.$$

279. The expression for the electromotive force of the couple AB between the temperatures t and t_1, is

$$E^{t}_{t_1}(AB) = \int_{t_1}^{t} \phi(t)\,dt = \int_{t_1}^{t} MN \times dt = \text{area } M_1MNN_1;$$

it is therefore represented by the area of the quadrilateral comprised between the curves AX and BX, and the ordinates corresponding to the two temperatures t_1 and t.

If, while the cold junction remains at a constant temperature t_1, the temperature t of the hot junction is increased, the electromotive force increases with the corresponding area, until the temperature attains the value t_n, which corresponds to the point of meeting of the two curves.

At this temperature t_n the thermoelectrical power of the two metals is zero; this is called the *neutral point.* When the temperature exceeds that of the neutral point, the electromotive force decreases, for it is now represented by the difference of the two triangular areas which have their apex at I; it becomes zero, as does the strength, at a temperature t_2 such that

$$\text{area } M_2IN_2 = \text{area } M_1IN_1.$$

Lastly, as soon as the temperature of the hot junction exceeds t_2, the electromotive force becomes negative and inversion takes place. Hence the temperature of inversion depends on the temperature of the cold junction.

280. The previous results become very simple when the curves AX and BX are straight lines. The figure M_1MNN_1 is then a trapezium, the surface of which has the value

$$P_1P \times \frac{M_1N_1 + MN}{2} = (t - t_1)\frac{\phi(t_1) + \phi(t)}{2} = (t - t_1)\,\phi\left(\frac{t + t_1}{2}\right).$$

We have further

$$\frac{\phi\dfrac{t + t_1}{2}}{t_n - \dfrac{t_1 + t}{2}} = \frac{\phi(t)}{t_n - t} = \text{const.} = a,$$

and, therefore,

$$E^{t}_{t_1}(AB) = a(t - t_1)\left[t_n - \frac{t_1 + t}{2}\right]$$

This expression is in conformity with the laws of Gaugain (272). If the straight lines AX and BX were parallel, we should have

$$E_{t_1}^{t}(AB) = A(t - t_1),$$

and the couple would have a uniform course.

281. Specific Heat of Electricity.—Suppose now that the variations of potential, to which electromotive force is due, are of two kinds; sudden variations, resulting from Volta's principle, and continuous variations connected with variations of temperature, and, like the former, capable of producing reversible calorific phenomena. It is clear that, if we designate the variations of the former kind by H, and the sum of the continuous variations which exist between the two points A and B of a conductor by $\int dh$, the value of the whole electromotive force will be

$$E = \Sigma H + \Sigma \int dh.$$

The variations of the second kind between two points M and M′ of the same metal, according to the law of Magnus, only depend on the temperatures t and t', and not at all on the intermediate resistance. We may then put

$$dh = f(t)\,dt = \sigma dt.$$

If the current is so small that the heating of the circuit on Joule's law may be neglected, the quantity of heat absorbed or developed in unit time in that portion of the circuit in which is produced the heat in question, by the passage of a current I, will clearly be expressed by

$$I dh = I f(t)\,dt = I\sigma dt.$$

The quantity σ is the variation of potential, and therefore the thermal work for unit current which corresponds to a variation of temperature equal to unity; it is a characteristic function of the nature of the conductor, but which varies with the temperature. Sir W. Thomson has given the name *specific heat of electricity* to this new physical quantity.

282. Electromotive Force of a Thermoelectric Couple.—This being admitted, let us consider a circuit (Fig. 69) formed of any number of metals $A_1, A_2, \ldots A_n$. Let $H_1, H_2, \ldots H_n$ be the sudden variations corresponding to the electromotive forces of contact; $\sigma_1, \sigma_2, \ldots \sigma_n$ the specific heats of electricity of the metals A_1,

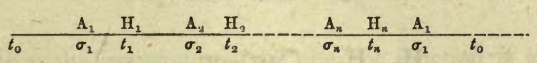

Fig. 69.

$A_2, \ldots A_n$; finally, let $t_1, t_2, \ldots t_n$ be the temperatures of the junctions, and t_0 the constant temperature of the external wire. The expression for the electromotive force of the circuit is

$$E = H_1 + H_2 \ldots + H_n + \int_{t_0}^{t_1} \sigma_1 dt + \int_{t_1}^{t_2} \sigma_2 dt \ldots + \int_{t_{n-1}}^{t_n} \sigma_n dt + \int_{t_n}^{t_0} \sigma_1 dt,$$

or, combining the two extreme integrals,

$$(3) \qquad E = \Sigma H + \int_{t_n}^{t_1} \sigma_1 dt + \int_{t_1}^{t_2} \sigma_2 dt + \ldots\ldots + \int_{t_{n-1}}^{t_n} \sigma_n dt.$$

Suppose that the circuit only consists of two metals A and A′, the electromotive forces of contact H_1 and H_2 are in general of opposite signs. Taking these signs into account, we shall have

$$(4) \qquad E = H_2 - H_1 + \int_{t_1}^{t_2} (\sigma' - \sigma) dt.$$

If the difference of temperature of the two junctions is infinitely small, we have $t_2 - t_1 = dt$, and the equation becomes

$$(5) \qquad \frac{dE}{dt} = \frac{dH}{dt} + \sigma' - \sigma = \phi_t(AA').$$

We have seen that for an infinitely weak current the circuit may be regarded as a reversible heat engine; we may therefore apply Carnot's theorem, and state that the algebraical sum of the quotients obtained by dividing the quantity of heat absorbed at a point, by the corresponding absolute temperature, is equal to zero.

Let T_2 and T_1 be the absolute temperatures between which the couple works, and T the absolute temperature of any given point of the conductor. At the hot junction the calorific work is $H_2 I$; at the cold junction $H_1 I$; on an element of the conductor taken between the temperatures T and $T + dT$, this work is $I\sigma dT$. Hence, suppressing the common factor, we shall have

$$\frac{H_2}{T_2} - \frac{H_1}{T_1} + \int_{T_1}^{T_2} \frac{\sigma' - \sigma}{T} dT = 0. \tag{6}$$

If the difference $T_2 - T_1$ is infinitely small and equal to dT, we may write

$$\frac{d}{dT}\left(\frac{H}{T}\right) + \frac{\sigma' - \sigma}{T} = 0. \tag{7}$$

or

$$\frac{1}{T}\frac{dH}{dT} - \frac{H}{T^2} + \frac{\sigma' - \sigma}{T} = 0,$$

and, lastly,

$$\frac{H}{T} = \frac{dH}{dT} + \sigma' - \sigma.$$

From equation (5), the second member of this equation is nothing but the thermoelectrical power $\phi(t)$ of the two metals; we have then

$$H = T\phi(t). \tag{8}$$

Thus, the electromotive force of contact of two metals, and therefore the Peltier effect at any given temperature, is equal to the product of the absolute temperature by their thermoelectrical power at the same temperature.

We deduce from this same equation

$$\frac{dE}{dt} = \phi(t) = \frac{H}{T},$$

and, therefore,

$$E = \int_{T_1}^{T_2} \phi\, dt = \int_{T_1}^{T_2} \frac{H}{T} dT. \tag{9}$$

283. The discussion of this formula leads to the different cases the examination of which we have anticipated (**276**). Let T_n be the temperature corresponding to the neutral point; for this point the thermoelectric power is zero, we have then

$$\phi(t_n) = 0 \quad \text{and} \quad H_n = 0.$$

So long as the temperature T_2 of the hot junction is lower than T_n, the function H_2 is positive, and the current cools the hot junction.

When $T_2 = T_n$, the heating effect is null at the hot junction. If the temperature T_2 is between the maximum temperature and that of inversion T_i, H_2 is negative; the current heats the hot and the cold junction at the same time.

Finally, if the hot junction is at a higher temperature than that of inversion, the current cools the two junctions.

According to Sir W. Thomson, and in conformity with Gaugain's experiments, the electromotive force of a couple may be empirically represented by the formula

$$E = a(T - T_1)\left[T_n - \frac{1}{2}(T + T_1)\right].$$

It follows from this that, for an infinitely small difference $d'T$ between the temperatures of the two junctions,

$$\frac{dE}{d'T} = a(T_n - T),$$

and, therefore,

$$H = aT(T_n - T).$$

The electromotive force of the couple and the electromotive force of contact between the two metals, expressed as a function of the temperature, will therefore both be represented by parabolas.

284. TAIT'S HYPOTHESIS.—Professor Tait arrived at the same result by assuming that the specific heat of electricity σ, characteristic of each metal, is proportional to the absolute temperature. Hence, denoting by k and k' the constants for each metal,

$$\sigma = kT, \qquad \sigma' = k'T.$$

In this case equation (7) becomes

$$\frac{d\left(\frac{H}{T}\right)}{dT}+(k'-k)=0,$$

and we deduce from it

$$\frac{H}{T}+(k'-k)T+C=0.$$

We have further, for the neutral point, $H_n=0$, or

$$(k'-k)T_n+C=0,$$

which gives, finally, putting $k'-k=a$ and replacing the absolute temperatures by ordinary temperatures,

$$H=(k'-k)T(T_n-T)=aT(t_n-t),$$

$$E=\int_{T_1}^{T_2}\frac{H}{T}dT=(k'-k)(T_2-T_1)\left[T_n-\frac{T_1+T_2}{2}\right],$$

$$=a(t_2-t_1)\left(t_n-\frac{t_1+t_2}{2}\right).$$

In this way we rediscover the empirical formulæ of Sir W. Thomson and of Gaugain.

The thermoelectrical power of the two metals is then

$$\phi(t)=\frac{dE}{dt}=\frac{H}{T}=a(t_n-t),$$

it will therefore be represented by a straight line as a function of the temperature.

Suppose we take the thermoelectrical powers in reference to the same metal for which k is zero, which, according to Le Roux's experiments, seems to be the case with lead, the equation reduces to

$$\frac{dE}{dt}=\phi(t)=k'(T_n-T).$$

The straight lines, which represent the thermo-electric powers of the different metals, are unequally inclined to the axis of temperatures; they cut this axis at the point corresponding to the temperature of the neutral point with the metal compared, and their inclination to the axis is the specific heat of electricity corresponding to each of the metals.

285.—Electrical Convection of Heat.—It was important to verify experimentally the hypothesis which serves as the basis for this theory—that is to say, the existence of changes of potential due to changes of temperature. The method employed by Sir W. Thomson consisted in establishing that, by the passage of a current, reversible calorific effects are produced analogous to the Peltier effect.

Let us consider a bar of metal (iron, for instance), the central part of which AA′ (Fig. 70), is kept at a constant temperature T, while the ends B and B′ are kept at 0°.

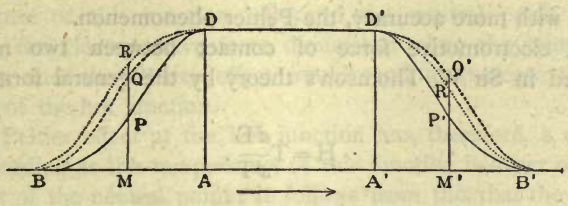

Fig. 70.

The distribution of temperatures is represented by the curves BDD′B′; so long as the current does not pass, the distribution is obviously symmetrical, and is represented by curves such as BPD and D′P′B′. The passage of a current produces two effects at each point; firstly, a heating regulated by the law of Joule; secondly, a disengagement or an absorption of heat produced by the fixed fall of potential corresponding to the difference of the temperature of two adjacent points.

If we only take into account Joule's law, the distribution of temperatures is still symmetrical, and may be represented by the dotted lines BQD, D′Q′B′.

The second effect is reversible with the direction of the current. If the current goes from left to right with the arrow, there will be a fresh heating in the anterior part of the wire BA, where the temperatures increase, and a cooling in the posterior part where they decrease. The distribution of temperatures is then dissymmetrical, and may be represented by the dotted curves BRD, and D′R′B′.

At two symmetrical points, M and M′, the difference of the final temperatures t and t', corresponds exactly to the Thomson effect. As the heating is greater in the part in front of the median region AA′, where the temperature is a maximum, there is a kind of *electrical convection of heat* in a direction opposite to that of the current, and this convection is proportional to the strength of the current.

Sir W. Thomson found in this way that for iron the electrical convection of heat is *negative* (that is to say, in the direction opposite to the current), and that the convection is *positive*, but far feebler, for copper.

M. le Roux extended these same observations to a great number of metals; he proved that the effect is proportional to the strength of the current, and ascertained that it is almost zero for lead, so that from this point of view lead is sensibly neutral.

286. NATURE OF THE PELTIER PHENOMENON.—We may now discuss, with more accuracy, the Peltier phenomenon.

The electromotive force of contact between two metals is expressed in Sir W. Thomson's theory by the general formula

$$H = T\frac{dE}{dT}.$$

If the course of the couple is uniform, we have, for the two temperatures T_1 and T,

$$E = A(T - T_1),$$

from which follows

$$H = AT.$$

The electromotive force of contact between two metals is then proportional to the absolute temperature, and the Peltier effect should follow the same law. At the temperatures of 25° and 100°, for instance, we shall have

$$\frac{H_{100}}{H_{25}} = \frac{273 + 100}{273 + 25} = 1 + \frac{75}{298} = 1 + \frac{1}{4}.$$

According to M. le Roux's experiments, the bismuth-copper couple exactly satisfies this condition.

A current which traverses a junction, heats it when it passes in one direction, and cools it when it passes in the contrary one. M. Becquerel showed that the direction in which there is cooling, is that of the current which would produce the artificial heating of the same junction. When a thermoelectrical current traverses a circuit, the variations of temperature produced at the junction by the current itself, tend then to become weaker, and we may say that their effect is to develop an electromotive force opposed to that which the current produces. That is a necessary condition; if it did not take place, an accidental current in a metallic circuit would produce a difference of temperatures between the junctions, which would go on increasing, and the current would maintain itself for an indefinite time.

In a circuit of two metals the hot junction of which is at a lower temperature than the neutral point, the electromotive force increases with the temperature, the Peltier effect tends then to diminish the temperature of this junction. Beyond the neutral point, on the contrary, the electromotive force diminishes when the temperature increases, and the Peltier effect would tend to increase the temperature of the hot junction.

The Peltier effect at the hot junction has, therefore, a different sign according as the temperature of this junction is lower or higher than that of the neutral point; it follows from this that the electromotive force of contact must have changed its sign at the neutral point. It is by analogous reasoning that Sir W. Thomson first showed this property of the neutral point, and deduced from it the necessity of the existence of electromotive forces in a homogeneous conductor at variable temperatures.

PART III.—MAGNETISM.

CHAPTER I.

PRELIMINARY.

287. On Magnets.—From the earliest times the name of *loadstone* has been given to certain natural ores which have the property of attracting iron filings; they consist of an oxide of iron whose chemical formula is Fe^3O^4. The various parts of a loadstone possess these properties of attraction to unequal extents: the filings attach themselves in preference to certain parts of the surface in the form of tufts.

These phenomena have an evident resemblance to those of statical electricity. The analogy, however, is not complete, and observation indicates essential differences between them; thus, the loadstone does not act indiscriminately upon all substances; the filings when attracted are not repelled after contact, and when once detached are not found to possess any new property. At each step in the course of this new study, we shall have to point out analogies and differences of this kind between the two orders of phenomena.

The loadstone, by mere rubbing, can magnetise steel—that is to say, can impart to it the property of attracting iron, and this without losing any of its own power. As steel bars, magnetised artificially, have a more regular form than loadstones, they are more convenient for investigation; experiment shows, moreover, that the phenomena are of exactly the same nature in the two cases.

288. Magnets—Natural and Artificial, Permanent and Temporary.—The general term *magnet* is given to all substances which have the property of attracting iron filings. *Natural* magnets are the pieces of magnetic ore found in nature; *artificial* magnets are pieces of steel, or specimens of iron more or less pure, to which the same properties have been imparted.

Some artificial magnets retain this new property when the cause which produced it has ceased to act; these are *permanent* magnets. Tempered steel is the body best suited for preparing permanent magnets, and is ordinarily employed in the form of long rods or bars.

Different kinds of cast and of wrought iron may also be powerfully magnetised by natural or by artificial magnets, but they lose most of their properties when the magnetising agent has ceased to act. In this way *temporary* magnets are obtained; and the term *residual* magnetism is applied to the comparatively feeble magnetisation which persists, at any rate for some time, in bodies which have been temporarily magnetised.

289. Magnetic and Diamagnetic Substances.—Until the present century, iron was the only substance which was known to be attracted by magnets; it was afterwards found that certain metals, such as nickel and cobalt, whose chemical analogies with iron are so remarkable, also possess these properties, but to a smaller extent. It has further been found, by means of very powerful magnets, that a great number of substances are also attracted by magnets, but the actions are incomparably weaker. The term *magnetic* is applied to all bodies which can be attracted by a magnet, and the term *magnetism* is applied to the whole of the phenomena to which magnets give rise, and, by extension, to the cause of these phenomena.

In 1778, Brugmans observed that a piece of bismuth is repelled by a magnet. The importance of this observation was disregarded, until Faraday's discovery that a certain number of other substances also possess this property. From the special way in which the experiment was made, Faraday applied the term *diamagnetic* to bodies which are repelled by magnets.

We may say, in short, that all bodies in nature are more or less susceptible of the action of magnets. They have been divided into two groups—*magnetic*, *paramagnetic*, or *positive* substances, which are attracted, like iron; and *diamagnetic* or *negative* substances, which are repelled by magnets, like bismuth.

290. Distribution of Magnetism in Magnets.—Poles.—We may obtain a bar magnetised regularly—a needle, for example—by rubbing it several times, and always in the same direction, with a piece of natural magnet, or with the same end of any artificial magnet.

When such a needle is placed in iron filings, the particles attach themselves more particularly to the ends of the needle, to a certain distance from them, and they stick end to end to each other, forming more or less abundant tufts.

Magnetic actions appear, therefore, to be concentrated at the ends of regular magnets. We shall call these ends the *poles* of the magnet, and shall afterwards define this term with more precision.

291. The Two Kinds of Magnetism.—The two ends of the magnet are not of the same kind; any magnet free to turn in a horizontal plane takes up a fixed direction in space. This direction is nearly north and south. When a magnet is displaced from this position it reverts to it when left to itself, and it is always the same end which points to the north. That end of the magnet which points to the geographical north is called the *north* pole, and that end which points to the south is the *south* pole. These poles may be marked once for all on permanent magnets.

292. Law of Magnetic Actions.—Magnets act on each other. The north pole of a magnet when presented to the north pole of another magnet repels it, but on the contrary attracts a south pole. In like manner two south poles repel each other. The phenomena are therefore analogous to those of electrical action: *Two poles of the same kind repel, and two poles of opposite kinds attract, each other.*

Hence there are four actions between two magnets which are near each other: two, which are repulsive, between poles of the same kind, and two attractive between different poles. If the magnets are very long as compared with their transverse dimensions, and are situated at a considerable distance in reference to these same dimensions, the action of each end may be considered as concentrated in a point. The reciprocal action of these two magnets consists, then, of four forces directed along the straight line which join in pairs the centres of action or the poles of the two magnets, and it is impossible in experiments to reduce them to a smaller number.

Yet, by causing the poles of two very long magnets to act on each other at distances, and in positions, such that the actions of the two other poles may be neglected, Coulomb experimentally established the law that *the attractive or repulsive actions between two poles are inversely as the square of their distance.*

It may, however, be remarked that nothing proves the existence of these elementary forces. Experiment shows, indeed, that the reciprocal action of the systems which constitute the magnets, and which are probably very complicated, may be reduced to attractive or repulsive forces directed along the right line joining the poles; but as one pole can never be separated from its fellow pole, the action of poles is a purely mental conception, advantageous no doubt in representing and calculating phenomena, but without any real existence as demonstrated by experiment. If it should happen

that other views as to the nature of elementary actions lead to the same conclusions, as regards the effects which we can measure (if, for example, we abandon the idea of action at a distance), we may consider these new views to be just as legitimate as previous ones.

293. MAGNETIC MASSES.—The action of two poles at a given distance depends on the special power of each of the magnets. Experiment shows that the actions exerted on the poles of two magnets by a given system are in a constant ratio; we may consider this ratio as being that of the magnetic masses of the two poles. It follows from this definition, that the action of any given system on a pole, is proportional to its magnetic mass; hence the reciprocal action of two poles is separately proportional to the mass of each of them—that is, to the product of their magnetic masses.

Calling these two masses m and m', the action f, of the two poles at the distance d, is

$$f = \phi \frac{mm'}{d^2},$$

ϕ being a coefficient which depends on the choice of unit mass. In order that this coefficient may be unity, we must take as *unit mass that of a pole, which, acting on an identical pole at unit distance, exerts a repulsion equal to unit force.* We have then

$$f = \frac{mm'}{d^2},$$

and the action is repulsive or attractive according as the poles are of the same or of opposite kinds.

If two poles of masses m and m', are connected with each other, the action of the system thus formed on a third pole M placed at distance d, which is very great in comparison with that of the two poles, is equal to

$$\frac{m\mathrm{M}}{d^2} + \frac{m'\mathrm{M}}{d^2} = \frac{\mathrm{M}(m+m')}{d^2};$$

it is proportional to the sum $m+m'$ of the two masses if the poles are of the same kind, and to the difference $m-m'$ if they are different. Magnetic masses can be added like algebraical quantities, and we may affix to them the signs + and − as we can to electrical

masses; we shall agree to give the sign + to the magnetic mass of a north pole, and the sign − to that of a south pole.

The action of two poles, expressed by the formula $f=\frac{mm'}{d^2}$, will be positive in the case of repulsion, the masses being of the same sign, and negative in the case of an attraction.

The law of elementary actions being the same as that for electrical phenomena, we may apply all the theorems relative to electrical potential, at any rate as regards fixed masses, and disregard, for the present, phenomena relative to conductors. The consideration relative to lines of force, to tubes and flows of force, are more particularly directly applicable to magnetism.

294. MAGNETIC FIELD.—A *magnetic field* is a space in which magnetic phenomena are produced. The *direction* and *strength* of the field at a point, are the direction and intensity of the force which would act on a positive magnetic mass equal to unity placed at this point.

295. DEFINITION OF POLES.—MAGNETIC AXIS OF A MAGNET.—We have assumed in the foregoing remarks, that the actions of a magnet reduce to that of two magnetic centres situate at the ends; this is the case, but then only approximately, when we are dealing with long cylindrical magnets, at a very great distance in reference to their transverse dimensions. Magnetic properties are really perceptible throughout the whole extent of the magnet, and only exhibit a very marked maximum near the ends. This is readily seen from the manner in which filings attach themselves to the magnet. We must admit, therefore, that in the magnet there is a series of magnetic masses, some positive and others negative, which are distributed according to a certain law, and the whole of which constitutes the total magnetic mass.

This being assumed, we may define more precisely what are called the poles of a magnet.

Let us suppose the magnet placed in a uniform magnetic field. The actions exerted by the field on the different points of the magnet are parallel to each other, and for each volume-element are proportional to the mass present there. All those which act on the positive masses are in the same direction: they have a resultant equal to their sum, and parallel to their direction, which is applied at the centre of mass, or the centre of gravity of the positive masses. The same is the case for negative masses, on which the field produces actions parallel to the preceding, but in the opposite direction. The magnet is submitted to the action of two parallel and opposite

forces, one applied at the centre of gravity of positive masses, and the other at the centre of gravity of negative masses. These two points of application are the *poles* of the magnet; the *magnetic axis* of the magnet is the line joining the two poles, and the *direction* of the magnetic axis is reckoned from the negative pole towards the positive one.

The magnet is evidently in stable equilibrium when its magnetic axis is parallel to the direction of the field, and pointing in the same way; equilibrium is unstable if these two directions are parallel but in contrary directions.

296. THE MAGNETIC MASS OF A MAGNET IS ZERO.—The vicinity of the Earth may be considered as a uniform magnetic field.

Experiment shows, in fact, that throughout a region whose extent is considerable in reference to the dimensions of the magnet, but small compared with the radius of the Earth, all magnets, when under the influence of the Earth alone, tend to assume the same direction.

Coulomb showed, moreover, that the action of the terrestrial field on any magnetised bar is purely directive—that it has neither vertical nor horizontal component; it has no vertical component, for the weight of a bar of steel is exactly the same before and after magnetisation; the horizontal component is also zero, for any magnet which can move in a horizontal plane has no tendency towards a motion of translation. *The two forces of opposite directions applied at the two poles are therefore equal, and constitute a couple.* From this follows this important conclusion—that in any magnet the sum of the positive masses is equal to the sum of the negative masses; in other words, *the total sum of the magnetic masses is zero.* We have then always $\sum m = 0$.

From this point of view the state of a magnet is comparable with that which a dielectric, or an insulated conductor, acquires by induction.

297. MAGNETIC MOMENTS.—Let m be the absolute value of the mass of each pole, and l the distance of the two poles; the product ml, of the mass by this distance, is called the *magnetic moment* M of the magnet.

This magnet may be represented by a straight line OA (Fig. 71) having for direction, the magnetic axis, and for length the numerical value of the magnetic moment M.

This mode of representation amounts to supposing that all the poles are identical, that their mass is equal to unity, for instance,

and to their being placed on the magnetic axis at a distance proportional to the magnetic moment of the magnet in question.

When a system formed of several magnets connected with each other is placed in a uniform field, the action of the magnet is reduced to a couple; accordingly, in order to estimate the total action, we may move all the magnets parallel to themselves, for instance, in such a manner that all the negative poles are superposed.

Consider two magnets represented by the right lines OA and OA′ (Fig. 71), and let G be the middle of the line AA′—that is to say,

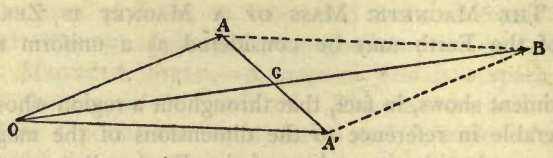

Fig. 71.

the centre of gravity of two masses equal to unity, placed at A and A′. The system is equivalent to a single magnet whose length is equal to OG and the masses equal to 2, or to a magnet of double length OB with masses equal to 1. The resultant magnet is thus represented by the diagonal of the parallelograms constructed on the right lines OA and OA′.

Magnetic moments of magnets may therefore be compounded as can forces. For any given system of magnets connected with each other, the resultant moment is represented by the straight line which closes the polygon constructed by adding, end to end, the moments of all the magnets.

The projection of this line on any given axis being equal to the sum of the projections of all the others, we see that the *magnetic axis of any given system, is the right line on which the sum of the projections of the separate moments of the magnets constituting the system is a maximum.*

In like manner, a magnet may be replaced by any given number of magnets, the resultant magnetic moment of which is equal to the moment of the proposed magnet—for instance, by the three projections of this magnetic moment on three rectangular axes.

When two magnetic systems are very distant from each other, their reciprocal action is equal to that of the resultant magnets, for each of the systems may be regarded as being situated in a uniform field produced by the other magnet.

298. Action of a Uniform Field on a Magnet.—If we consider a magnet whose moment M equals ml, situated in a uniform field whose strength F makes the angle θ with the axis of the magnet, the moment of the couple produced by the action of the field is equal to $Fml \sin\theta$ or $FM \sin\theta$. This is the moment of the couple which would tend to turn the magnet about a straight line perpendicular to the magnetic axis and to the force of the field.

If the magnet is movable about a given axis, the couple of rotation only depends on the projections M_1 and F_1 of the magnetic moment and of the strength of the field, on a plane perpendicular to the axis, for the projections on the axis are without influence. If θ_1 is the angle of the directions of M_1 and of F_1, the moment of the couple is equal to $F_1M_1 \sin\theta_1$.

Let, generally, a, b, c and α, β, γ denote respectively the cosines of the angles which the directions of F and M make with three rectangular axes, and let us replace these magnitudes by their projections on the three axes. The moment of the couple which tends to turn the magnet about the axis of z is

$$Z = Fb.M\alpha - Fa.M\beta = FM(b\alpha - a\beta).$$

We shall have, similarly, for the other axes,

$$X = FM(c\beta - b\gamma),$$
$$Y = FM(a\gamma - c\alpha).$$

The product FM is sometimes called the moment of the action of the field upon the magnet; it is the moment of the couple which would be produced if the magnet were perpendicular to the direction of the field. As a particular case, if T is the intensity of the terrestrial field, the product TM will be the moment of the terrestrial action on the magnet.

299. Astatic Systems.—Take the particular case of two magnets (Fig. 72) whose magnetic moments are OA and OA', the

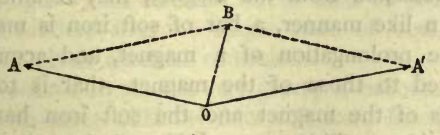

Fig. 72.

resultant moment is the diagonal OB of the parallelogram drawn on OA and OA'. If the moments OA and OA' are equal and exactly

opposite, the resultant moment is null, and the equilibrium is neutral in any position whatever in a uniform field; such a system is said to be *astatic*.

If the moments OA and OA′ are almost equal and make an angle nearly equal to 180°, the resultant OB is very small and is directed sensibly in the direction of the line bisecting the angle AOA′; it is therefore perpendicular to each of the needles. Thus, when two magnetised needles, forming a *quasi astatic* system, are in a uniform magnetic field, they are in stable equilibrium in a direction at right angles to the force of the field.

This is exactly the case of magnetic needles used for certain galvanometers. The system is so much the more nearly astatic the more nearly the direction of the free needles is to being perpendicular to the magnetic meridian.

300. Magnetic Polarity.—Rupture of a Magnet.—When a magnetic needle is broken, each of the portions becomes a complete magnet having two equal poles of opposite kinds, and the phenomenon can be repeated indefinitely as far as we can carry the division by mechanical means. This is a fact of prime importance in the theory of magnetism: it proves in the first place that it is impossible to get an independent mass of negative or positive magnetism which is not associated with an equal mass of the opposite kind; and further that magnetism is an essentially molecular phenomenon. We are led to admit that magnetism is due to a kind of polarization of ponderable molecules, each of which is a small magnet with its two poles exactly on the terminal faces.

301. Induced Magnetisation.—The tufts of iron filings which remain adhering to a magnet prove that each particle of filing has itself become converted into a small magnet. The number of grains in direct contact with the magnet is relatively very small; the others, attached in succession to each other and to the first, form chains where the particles are united by their poles of contrary names. The magnetisation acquired by these filings is transient; as soon as they are detached from the magnet, they resume their original neutrality. In like manner, a bar of soft iron is magnetised when placed in the prolongation of a magnet, and acquires two poles similarly placed to those of the magnet—that is to say, the two adjacent ends of the magnet and the soft iron have magnetisms of opposite signs. This magnetised soft iron may in turn act similarly on a second piece, and so on. As soon as the original magnet is removed, the magnetisation of the first bar of soft iron and of all those which follow it disappears more or less, and all the

actions, which they exert on each other, disappear at the same time.

More generally, when any magnetic body is placed in a magnetic field, it becomes itself a magnet. This is a *magnetisation by influence* or *induced magnetisation.*

The axis of magnetisation at each point is parallel to the direction of the resultant force. This resultant arises from the action of the field and of that which is produced by the induced magnetism itself. If the body in question is infinitely small, the magnetisation is exactly parallel to the force of the field at the point in question.

This conclusion also follows, that the action of a magnet is null on a neutral body, and that any action exerted by magnets on magnetic bodies, is preceded by a magnetic induction on the latter.

Here again we see the analogy of this phenomenon with that of electrostatic induction, and particularly the induction in dielectrics.

The magnetism thus induced does not depend merely on the strength of the field, but also on the nature of the substance in question; magnetisation, which is very powerful with pure iron and nickel, is far feebler with all other magnetic substances.

302. Soft Iron.—Coercive Force.—Iron is said to be absolutely *soft* if, after having been placed in a very powerful magnetic field, it loses its magnetisation when it is withdrawn from it. Soft iron, in the magnetic sense, is also soft in the ordinary meaning of the word; it may be easily bent, worked, and it has but little elasticity. Conversely, ordinary iron is not soft in the magnetic sense of the word; when it is impure, or has undergone mechanical changes, it remains more or less magnetised, and this property is designated by the somewhat barbarous term *coercive force.* A specimen of iron has so much the greater coercive force the greater is its quantity of residual magnetism; at the same time, this kind of iron is more readily magnetised by induction. Coercive force is therefore a property analogous to friction. Within certain limits it opposes the changes which external forces tend to produce in magnetisation, and hinders any one single state of equilibrium from corresponding to given external conditions.

The coercive force in steel is very great; it becomes magnetised by induction with more difficulty than soft iron, but it retains the magnetisation, once acquired, so much the better. The magnetic qualities of steel vary with the composition of the metal and with its mode of preparation; they depend greatly on the manner in which the tempering has been effected, as well as on the degree of annealing

to which the bar has been afterwards subjected. The harder the steel, and the more brittle the temper, the greater is the coercive force.

We have hitherto implicitly assumed that the magnetisation of a magnet is invariable, and is independent of the forces to which the magnet is subjected; but this kind of *magnetic rigidity* is a limiting case which is never realised with perfect completeness. When a magnet placed in a strong magnetic field is in its normal position of equilibrium, its magnetisation slightly increases; it diminishes on the contrary if it is in the opposite direction. The variations thus produced are generally feeble, and usually transient like those of soft iron in the same circumstances; these variations may ordinarily be neglected in the case of powerfully magnetised steel bars placed in a magnetic field of no great strength—like that of the earth, for instance.

303. INFLUENCE OF TEMPERATURE.—Heat acts also on the magnetism of magnets. A moderate increase of temperature diminishes the magnetisation, but only temporarily, and the magnet resumes its original magnetisation with its original temperature. Within the ordinary variations of the surrounding temperature, the effects produced are sensibly proportional to these variations, so that if M_0 and M_t are the magnetic moments of a magnet at the temperature of zero, and of t degrees, we have the ratio

$$M_t = M_0(1 - at),$$

the coefficient a depending on the nature of the steel.

A greater degree of heating, above 100° for instance, produces a definite enfeeblement of the magnetisation, and a bar of steel heated to bright redness has usually lost all traces of magnetisation when it returns to the ordinary temperature.

A rise of temperature produces analogous effects on the magnetic properties of soft iron. At the ordinary temperature the magnetisation induced in iron by a given field, changes but little with variation of temperature, but beyond 100° the diminution of induced magnetism becomes very rapid. At a temperature beyond red heat, iron no longer possesses the power of being attracted by magnets; at that temperature it is not even magnetic.

304. ON MAGNETIC FLUIDS.—The physicists of last century, more especially Æpinus and Coulomb, attempted to explain magnetic phenomena by a hypothesis analogous to that of electrical fluids.

From this point of view we must attribute to the fluids, to the magnets, and to magnetic substances, a certain number of properties by which all the experiments may be explained.

We assume, then, the existence of two imponderable magnetic fluids, consisting, like the electrical fluids, of molecules which act by repulsion on the molecules of the same fluid, and by attraction on molecules of a different kind, these reciprocal actions being inversely as the square of the distance. The combination of these two fluids in equal quantities has no action on external bodies, and constitutes what may be called the neutral fluid.

In virtue of the phenomena of induced magnetisation, we must assume that the neutral fluid exists in almost unlimited quantity in magnetic bodies, and is divided into two distinct fluids under the influence of the magnet.

Since permanent or temporary magnets are always complete, whatever may be their dimensions, we must assume also that the fluids present in an element of volume never quit it to pass to an adjacent element, so that the separation of these fluids is confined to the extent of each molecule.

Finally, no internal force opposed to the directive actions of the magnetic fluids, hinders their separation or their reunion in soft iron. In cast iron and in steel, on the contrary, there is a special resistance (a kind of friction called coercive force), which restricts the magnetisation by induction, and then hinders the recombination of the fluids when the external force has disappeared.

It is not surprising that the theory of fluids, with all the accessories which are only arbitrarily connected with it, can explain the phenomena; in these conditions the agreement of experiment with theory affords no argument in favour of the exactitude of the hypothesis; and we shall make no further use of it.

305. DEFINITION OF TERRESTRIAL MAGNETIC ELEMENTS.— The magnetic field which surrounds the earth, and which may be called the terrestrial field, is sensibly uniform throughout a space of small dimensions as compared with that of the terrestrial radius; but the direction and the intensity of the force vary from one point to another. In fact the force in any one place changes in magnitude and direction in the course of time; we shall disregard for the moment these variations which are feeble, and shall suppose we are considering the magnetic state of the globe at a definite time.

The magnetic axis of a magnet suspended freely by its centre of gravity, and withdrawn from any other action than that of the terrestrial magnetic field, would, when in equilibrium, assume the

direction of the terrestrial forces. In our country this direction is almost north and south, and it makes a considerable angle with the horizontal line, the north pole pointing downwards.

The *magnetic meridian* in any place is the vertical plane passing through the direction of the earth's magnetic force.

The *declination* is the angle which the magnetic meridian makes with the astronomical meridian; the declination is said to be *west* when the north pole of a free magnet turns to the west of the magnetic meridian which passes through its centre; it is *east* if this north pole is to the east of the meridian.

The *inclination* is the angle which the earth's force makes with its projection on a horizontal plane.

Let

D be the declination,
I the inclination,
T the strength of the earth's field,
H the horizontal component = T cos I,
Z the vertical component = T sin I.

A magnetised needle, movable about a vertical axis, will only obey the horizontal component of the earth's force, and will place itself so that its axis of magnetisation is in the magnetic meridian. If we move it out by an angle δ, the moment of the couple which tends to bring it back, has the value

$$HM \sin \delta;$$

M being the magnetic moment of the needle. It is proportional to the sine of the angle of deviation. This result has been verified by the very accurate experiments of Coulomb by means of his torsion balance.

If the needle is suspended freely by its centre of gravity, or is movable about a horizontal axis passing through this point, and perpendicular to the magnetic meridian, the direction of the magnetic axis, when the needle is in equilibrium, is the direction of the earth's force itself; the angle which its magnetic axis then makes with the horizontal measures the inclination.

Let us now suppose that the horizontal axis of rotation makes an angle α with the perpendicular to the magnetic meridian. We may replace the horizontal component H by its two projections, the one $H \sin \alpha$ parallel to the axis of rotation, and other $H \cos \alpha$ perpendicular to this axis. The needle only obeys the two forces Z and

$H\cos\alpha$ situate in the plane which it describes; the value of the resultant of these two forces is

$$T_1 = \sqrt{Z^2 + H^2\cos^2\alpha},$$

and the needle, in its position of equilibrium, makes with the horizontal an angle i, defined by the equation

$$\cot i = \frac{H\cos\alpha}{Z} = \cot I\cos\alpha.$$

The angle i is the apparent inclination in the vertical plane which is at an angle α with the magnetic meridian. If we have $\alpha = \frac{\pi}{2}$, it follows that $i = \frac{\pi}{2}$ and the needle is vertical. If the angle α changes by $\frac{\pi}{2}$, we have $\cos\left(\alpha \pm \frac{\pi}{2}\right) = \mp\sin\alpha$, and the value of the new inclination is

$$\cot i' = \mp\cot I\sin\alpha.$$

From these two equations we have,

$$\cot^2 i + \cot^2 i' = \cot^2 I,$$

a formula frequently used in determining the inclination.

If the needle is loaded with an accessory weight, or, what is the same thing, if the axis of rotation does not pass through the centre of gravity, the direction of the equilibrium is modified.

Suppose, as a particular case, that a weight p, at a distance d from the axis of rotation, keeps the needle horizontal in a certain plane; the moment of the magnetic couple is reduced then to the moment of the vertical component, and the condition of equilibrium is

$$pd = MZ = MT\sin I.$$

This condition is independent of the azimuth of the vertical plane in which the needle moves; the counterpoise which makes the needle horizontal in one plane, would make it horizontal in all planes. Hence, placing a needle on a vertical pivot, we may counterpoise it so that it is always horizontal; but the weight of the counterpoise depends on the vertical component, and this ought to be modified if we wish to use the needle in other latitudes.

306. Distribution of Terrestrial Magnetism.—The elements of terrestrial magnetism, strength, declination, and inclination are not the same at the different points of the earth. These elements vary as a function of the geographical co-ordinates according to very complicated laws; but if we are content with a first approximation, the variations may be formulated in a very simple manner.

The magnetic meridian at any one place cuts the surface of the globe along a great circle; all points of this great circle have the same plane for magnetic meridian.

All magnetic meridians intersect along the same diameter; this diameter is the *magnetic axis* of the earth; the points where it cuts the surface have been named, though incorrectly, *magnetic poles.* The magnetic axis makes an angle of about 15° with the axis of rotation of the earth.

It is evident that the declination varies from one point to another on the same magnetic meridian. The only exception is the meridian which, passing both through the magnetic axis and the terrestrial axis, is identical with the geographical meridian; for all corresponding points the declination is null. On one side of this great circle, the north pole turns to the west and the declination is west; on the other, it turns to the east and the declination is east.

The great circle perpendicular to the magnetic axis is called the *magnetic equator.* In all points of the magnetic equator, the earth's force is horizontal and the inclination zero. On either side the inclination increases to the magnetic poles where it is 90°: in the northern hemisphere, the north pole dips downwards; and in the southern hemisphere, the south pole.

307. Hypothesis of a Terrestrial Magnet.—Biot tried if it were possible to represent the magnetic condition of the globe, and the variation of the magnetic elements on its surface, by the hypothesis of a central magnet in the direction of the magnetic axis; he found that the results of calculation agreed the better with the observations the smaller was the distance of the poles of this imaginary magnet.

If we thus replace the earth by a magnet which is infinitely small as compared with the radius—that is to say, by two equal masses of opposite signs which are very near each other, we know (153) that at the latitude λ, counting from the magnetic equator, the inclination is given by the equation

$$\tan I = 2 \tan \lambda.$$

The intensity of the force T at any point of the surface may be expressed as a function of the force at the equator T_e by the formula

$$T^2 = T_e^2(3\sin^2\lambda + 1).$$

At the magnetic pole the intensity is

$$T_p = 2T_e;$$

it is therefore twice as great as at the equator.

These two formulæ are at any rate approximately in agreement with observations made at a given moment over the whole surface of the earth.

The absolute magnetic moment of the earth ϖ may be obtained simply by means of the equation

$$\varpi = T_e R^3 = \frac{TR^3}{\sqrt{3\sin^2\lambda + 1}},$$

in which R represents the earth's radius.

The hypothesis of a terrestrial magnet was introduced into science by Gilbert. The pole in the southern hemisphere received the name of austral pole, and is evidently of the same kind as the north pole of magnets; the pole situated in the northern hemisphere is of the same kind as the south pole of magnets.

This conception of a terrestrial magnet has also led to the designation of austral pole being applied to that pole of the needle which turns towards the north, and of boreal pole to that which turns towards the south; in the theory of fluids, in spite of the contradiction, we may say that a north pole contains austral fluid and a south pole boreal fluid. But it is better to abandon the expressions *austral* and *boreal*, which may give rise to misconceptions, and to call, as we have done, positive magnetism that which corresponds to the north pole of magnets, and negative magnetism that of the south pole.

308. The hypothesis of an infinitely small central magnet is only one of the forms under which the earth's magnetism may be represented; it is even that which is least probable, seeing that the undoubtedly very high temperature of the centre of the earth is incompatible with the existence of bodies strongly magnetised.

We know, for instance, that two superficial hemispherical layers equal and of opposite signs, distributed so as to produce a constant

force at any point in the interior, and which we have called layers of gliding (157), will produce on the exterior the same effects as two infinitely near masses. The values of the densities of these layers at the poles will be

$$\sigma_0 = \frac{3}{8\pi} T_p = \frac{3}{4\pi} T_e,$$

and, at a point of the magnetic latitude λ,

$$\sigma = \sigma_0 \sin \lambda.$$

From this point of view, the earth must be considered as covered with two magnetic layers, the one negative in the northern hemisphere and the other positive in the southern one, the density at each point being proportional to the sine of the magnetic latitude.

The total mass of each of the layers is expressed by

$$\pi R^2 \sigma_0 = \frac{3}{4} R^2 T_e;$$

it may therefore be easily calculated if we know the absolute value of the force at the equator.

We shall see, in the sequel, that there are other modes of representing terrestrial magnetism; the infinitely small central magnet, which of itself is inadmissible, is really the very simple mathematical expression of several equivalent states, which are quite compatible with the known properties of magnetic substances.

309. VARIATIONS OF TERRESTRIAL MAGNETISM.—The elements of the earth's magnetism also undergo changes with the time; one kind are purely accidental, while others have a well-marked periodical character. The variations of long periods, which are called secular variations, may be represented, as a first approximation, by a rotation of the magnetic axis about the earth's axis, a rotation in virtue of which the magnetic axis should describe from east to west a circular cone of about 30°.

As the earth's magnetic pole, which at present is in New South Wales in 100° W. longitude, was in 1660 near the North Cape in 20° E. longitude, we see that the period of complete revolution is about 800 years. The declination at Paris, which at first was east, was null in 1666; since this time it has been west, and went on increasing until 1824; it is now decreasing and will be null in 2050,

if the phenomenon continues to follow the same course; the magnetic pole will then be on the other side of the north pole in reference to us. Since 1666 the inclination in Paris has been continually decreasing: it will attain a minimum when the declination is null.

The variations with a short period seem to be connected with the apparent motion of the sun, of the moon, etc., and are governed by laws which at present are not well known.

The mean values of the declination, for instance, have in one and the same place a well-marked daily oscillation with two maxima and two minima. The amplitude of the excursion of the magnet is far greater during the day than during the night, and the time of the extreme variations is very different according to the positions of the stations. Thus, while the average maximum westerly deviation over a whole year is at 9 a.m. at Hobart Town (Tasmania), Batavia, the Cape, and St. Helena, this time corresponds to the maximum easterly variations in the northern hemisphere. The maximum westerly excursion is at 1 p.m. at Toronto (Canada), London, and Paris; at 2 o'clock at St. Petersburg; at 3 at Nertchinsk and Pekin. The hours of these maxima and minima vary, moreover, with the seasons. The other magnetic elements, inclination and components of the force, present analogous oscillations.

By eliminating the mean daily variation from observations relative to the various magnetic elements, we may refer them to the lunar day, and we thus find a regular variation in the residual effects.

There is, further, an annual periodical variation.

Finally, the accidental variations themselves, which seem to be produced simultaneously over a great extent, if not over the whole surface of the globe, and which are ordinarily known as perturbations or magnetic storms, appear also to occur in certain annual or secular periods as regards their main effects. These perturbations are directly related to the phenomenon of the aurora borealis, and are accompanied by accidental currents in telegraph wires.

CHAPTER II.

CONSTITUTION OF MAGNETS.

310. MAGNETIC FILAMENTS.—The experiment of breaking a magnetised bar demonstrates this central fact that any volume element of a magnet is itself a complete magnet, having in its then state a magnetic axis and a definite moment. We say in its then state, for it is clear that if the volume element instead of being conceived as separated from the surrounding medium, were really detached, it would no longer retain the same state as when it formed part of the general mass. Let us consider two molecules placed end to end, and only touching with their opposite poles; if they are equally magnetised, the action for any external point would reduce itself to that of its two free ends. In like manner, if a series of molecules equally magnetised are placed end to end, all the magnetic axes being arranged on the same line, the external action of the linear magnet thus constructed still reduces to that of its two ends, each intermediate point giving rise to equal and contrary actions which neutralise each other. Such a system of magnetised particles constitutes a *uniform magnetic filament.*

311. FREE MAGNETISM.—But if the magnetisation in this line of molecules is variable, at each point there will be a certain quantity of apparent or *free* magnetism, equal to the differences of the magnetic masses of two adjacent molecules in contact. If we suppose, for instance, that the magnetisation diminishes from the middle of the filament to the end, we see that on one of the halves of the linear magnet, there will be an excess of positive magnetism distributed according to a certain law, and on the other half an equal excess of negative magnetism. The magnetic filament thus constructed is no longer uniform, but it is evident that we can regard it as the resultant of the juxtaposition of uniform filaments of different lengths.

312. Uniform Magnet.—A magnet of finite dimensions, which is formed of identical filaments placed parallel to each other, might be called a *uniform* magnet; the poles of the elementary filaments being placed at the ends, on the surface of the body, it will be seen that the action of the whole magnet would reduce to that of two magnetic layers, distributed on the surface according to a simple law.

313. Any Given Magnet.—At a point P within any given magnet the magnetic axis has a determinate direction, and this direction varies continuously; hence, inside a magnet we may draw lines tangential at every point to the magnetic axis, and we may imagine magnetic filaments directed along these lines of magnetisation.

The magnet would thus be subdivided either into non-uniform magnetic filaments closed or terminating at the surface, or into uniform filaments, some closed, and others terminating on the surface, and others, lastly, terminating in the interior. So long as no hypothesis is made as to the form of the filaments, this conception is a pure and simple translation of facts, and has nothing hypothetical. It leads to considering any given magnet as formed of a magnetic layer distributed on the surface, and of magnetic masses disseminated throughout the interior. We may accordingly consider a surface density of magnetisation and a volume density.

The density of the free magnetism at a point is the limit of the ratio of the magnetic mass contained in a volume element taken about this point to the volume itself; the surface density is the quotient of the quantity of magnetism which exists on an element of surface about this point, by the area of the element.

314. Potential of a Magnet.—This being admitted, it is clear that the value of the potential of the magnet at any external point P will be

$$(1) \qquad V = \int \frac{\sigma}{r} dS + \int \frac{\rho}{r} dv.$$

In the first integral, which will extend to the entire surface of the magnet, σ denotes the surface density on the surface element dS, whose distance from the point P is equal to r. The second integral must be extended to the whole volume of the magnet, ρ denoting the volume density of magnetism in the element dv at a distance rv from the point P. These densities ρ and σ may be considered as those of a particular fluid.

As the sum of the magnetic masses is always zero for any given magnet, we have the equation of condition

$$0 = \int \sigma d\mathrm{S} + \int \rho dv.$$

The components of the magnetic force at the point P are

$$\mathrm{X} = -\frac{\partial \mathrm{V}}{\partial x},$$

$$\mathrm{Y} = -\frac{\partial \mathrm{V}}{\partial y},$$

$$\mathrm{Z} = -\frac{\partial \mathrm{V}}{\partial z};$$

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and the value of the force itself is

$$\mathrm{F} = \sqrt{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2} = \sqrt{\left(\frac{\partial \mathrm{V}}{\partial x}\right)^2 + \left(\frac{\partial \mathrm{V}}{\partial y}\right)^2 + \left(\frac{\partial \mathrm{V}}{\partial z}\right)^2}.$$

These formulæ are general, and apply not only to points on the exterior but also to points in the interior of the magnet.

It can be shown that, as in electricity, the sum $\Delta \mathrm{V}$ of the three partial secondary differentials of the potential is equal to zero for any external point, and to $-4\pi\rho$ for any point inside the magnetised bodies.

The fundamental equations are the same as for statical electricity. Hence, without a fresh demonstration we may apply the theorems already established, provided that these theorems do not depend on the properties of conductors, and that, on the other hand, the coercive force does not come into play.

315. A MAGNET IS EQUIVALENT TO A MAGNETIC SURFACE.—We may demonstrate directly Poisson's theorem, that the action of a magnet on an external point is equivalent to that of a fictive layer of a total mass equal to zero, distributed along the surface according to a certain law.

Suppose that, as a matter of fact, the masses in question are fixed electrical masses, and that the magnet is covered with an infinitely thin conducting surface in contact with the ground. On the inner surface of the conductor a layer of opposite sign to the internal

masses will be developed by induction; as the force and the potential are now everywhere null on the exterior, the layer induced has, on any external point, a potential equal and of opposite sign to that of the original masses. A layer equal and of the opposite sign distributed on the external surface according to the same law, will produce a potential equal and of the same sign as those of the system in question, and will form therefore a system equivalent for external points. This conclusion obviously applies to magnetism, the two kinds of masses obey the same elementary law.

The external actions exerted by the magnet enable us to calculate the density of this fictive surface layer, but will teach nothing respecting the real distribution of magnetism. In order to investigate this distribution, it would be necessary to determine experimentally the forces which act on the interior of the magnet, and make cavities into which test needles could be introduced; but the withdrawal of a mass, however small, modifies the force in the cavity, for the adjacent masses are suppressed, whose effect cannot then be neglected. The magnet is equivalent then to two fictive layers, one on the outer and the other on the inner surface of the cavity; the sum ΔV of the three secondary differentials of the potential is become null in this region, while it was originally equal to $-4\pi\rho$.

316. Poisson's Theory.—Hitherto we have made no hypothesis as to the manner in which magnetic masses are distributed in the magnetised substance. To establish the theory of magnetisation by induction, Poisson considers a magnetised body as made up of magnetic particles disseminated in a medium impervious to magnetism. These particles are spherical and equidistant if the body is isotropic and homogeneous; each of them contains equal quantities of positive and negative fluids, part in the neutral state in the interior, and part in the free state on the surface. The magnetic moment of each particle, of volume u, may be represented by uq, the factor q depending on the degree of magnetisation. If we consider a volume dv, which is very great compared with the dimensions of the particles, but infinitely small in reference to the dimensions of the magnet, all the particles which it contains will have their magnetic axes sensibly parallel, and the magnetic moment of the volume element will be the sum of the moments of the particles. Calling h, as we have already done (167), the ratio of the space occupied by the particles to the total volume dv, the total volume of the particles contained in this element is proportional to hdv, and its magnetic moment will be $hqdv$. This element will act on any point at a finite distance like

an infinitely small magnet, or like the system of two infinitely near, equal, and contrary masses (151). The magnetic moment of the magnet for unit volume is equal to hq.

The value of the ratio h varies with different magnetic bodies, and for the same body the value of q at each point depends on the degree of magnetisation; external actions increase or diminish with the product hq. In bodies which have no coercive force, nothing prevents the movement of fluids in the interior of a magnetic particle; equilibrium can only exist when the resultant of all the forces, internal as well as external, is zero for every point of the molecule; on the contrary, in a body endowed with a certain coercive force, which acts like friction, it is sufficient if this resultant be less than the value given for the coercive force.

Poisson's theory is not bound up with the hypothesis of two fluids, but it is more difficult to free it from this particular conception of the structure of magnetic media.

317. Sir W. Thomson's Theory.—We shall prefer to explain the theory of magnetism in the form given to it by Sir W. Thomson. This theory agrees with that of Poisson in its essential results, but it has the advantage of being independent of the idea of fluid, and of any hypothesis on the constitution of the medium, so that it seems to be in closer agreement with experimental facts. The fundamental notion is to consider any given portion of a magnet as being a complete magnet, defined by the direction of the axis and by its magnetic moment—that is to say, as an infinitely small magnet having masses $+m$ and $-m$ at its ends, a length ds, and therefore a magnetic moment equal to mds.

318. Intensity of Magnetisation.—That being admitted, the term *intensity of magnetisation* I at a point, is the quotient of the magnetic moment of a volume element by the volume itself—in other words, the value of the moment for unit of volume.

We shall have thus

$$\text{(2)} \qquad \mathrm{I} = \frac{mds}{dv}.$$

This intensity of magnetisation I, represents the product hq in Poisson's theory.

The intensity of magnetisation is a geometrical magnitude defined, like a force, by its direction, which is the magnetic axis of the volume element, and by its numerical value; it will therefore be represented at every point by a straight line of given direction and magnitude.

All magnetic phenomena may be expressed as a function of this quantity alone.

319. EXPRESSION FOR POTENTIAL.—Let I be the intensity of magnetisation at a point M of the magnet whose co-ordinates are x, y, and z. If the intensity of magnetisation makes, with the axes, angles whose cosines are λ, μ, ν, its components A, B, C, along the axes will be expressed by

$$A = I\lambda,$$
$$B = I\mu,$$
$$C = I\nu.$$

The magnetic moment of a volume element is $mds = Idv$. Its potential at a point P at a distance r along a right line, making an angle θ with the direction of the magnetic axis (that is to say, with the direction of the strength of magnetisation), is equal to (151)

$$dV = Idv\frac{\cos\theta}{r^2}.$$

This potential may be regarded as the sum of the potentials dV_a, dV_b, dV_c, due to the three components A, B, C, of the magnetisation. If we denote by δ the angle which the right line MP makes with the axis of x, and by ξ, μ, ζ, the co-ordinates of the point P, we have,

$$dV_a = \frac{Adv\cos\delta}{r^2} = \frac{Adv(\xi - x)}{r^3}.$$

On the other hand, the equation

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = r$$

gives

$$(\xi - x) = -r\frac{\partial r}{\partial x},$$

and, therefore,

$$\frac{\xi - x}{r^3} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = \frac{\partial\frac{1}{r}}{\partial x}.$$

From these we deduce

$$dV_a = Adv\frac{\partial\frac{1}{r}}{\partial x} = A\frac{\partial\frac{1}{r}}{\partial x}dxdydz.$$

In like manner we shall have

$$dV_b = B\frac{\partial \frac{I}{r}}{\partial y}dxdydz,$$

$$\partial V_c = C\frac{\partial \frac{I}{r}}{\partial z}dxdydz.$$

The potential of the entire magnet will be obtained by extending these expressions to the whole volume, which gives

$$(3) \qquad V = \int\int\int \left(A\frac{\partial \frac{I}{r}}{\partial x} + B\frac{\partial \frac{I}{r}}{\partial y} + C\frac{\partial \frac{I}{r}}{\partial z} \right) dxdydz.$$

The potential at the point P is thus expressed as a function of the distance r of this point from the different elements of volume of the magnet, and of the intensity of the magnetisation.

Each of the terms of which the second member of the equation (3) is composed contains a factor which is an exact differential, and may be integrated by parts; we then obtain

$$V = \int\int \frac{A dydz + B dzdx + C dxdy}{r} - \int\int\int \frac{I}{r}\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz,$$

the former integral should be extended to the whole surface, and the second to the volume of the magnet.

Let I be the intensity of magnetisation at a point of the surface S (Fig. 73), and θ the angle which its direction makes with the perpendicular; α, β, γ, the cosines of the angles of the perpendicular

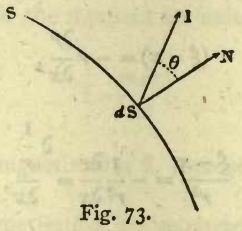

Fig. 73.

with the axes; lastly dS, an element of surface at the point in question, we have

$$I dS \cos\theta = I dS(\alpha\lambda + \beta\mu + \gamma\nu)$$
$$= I\lambda . \alpha dS + I\mu . \beta dS + I\nu . \gamma dS.$$

The products $I\gamma$, $I\mu$, $I\nu$ are the components of the magnetisation, and αdS, βdS, γdS, the projections of the element of surface on the co-ordinate planes. We have then

$$I ds \cos\theta = A dydz + B dzdx + C dxdy,$$

and the expression for the potential becomes

$$(4) \quad V = \int \frac{I\cos\theta}{r} dS - \iiint \frac{1}{r}\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz.$$

The formulæ (1) and (4) represent the same potential; if they are identified it will be seen that the surface density, and the volume density of magnetism, may be expressed as a function of the intensity of the magnetisation in the following manner.

$$(5) \quad \sigma = I\cos\theta = A\alpha + B\beta + C\gamma,$$

$$(6) \quad \rho = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right).$$

Hence, *the surface density is the resolved part of the intensity of magnetisation in the direction of the perpendicular to the surface drawn outward.*

The volume density is equal and of opposite sign to the sum of the partial derivatives of the components of magnetisation referred to three axes.

The quantities ρ and σ, which represent the densities of a fluid on Coulomb's hypothesis, may be regarded as purely mathematical quantities. They are two symbols defined by the equations (5) and (6); for the sake of brevity the name of densities will be retained, without attaching to this word a literal meaning.

We may observe that Poisson's equation, relative to secondary differentials, becomes in the present case

$$\Delta V = -4\pi\rho = 4\pi\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right).$$

For an external point, the second member is identical with zero since the strength of magnetisation is constant and equal to zero.

320. UNIFORM MAGNETS.—Let us consider the particular case in which the magnetisation is *uniform*, that is to say, in which the

strength of magnetisation is constant in magnitude and direction throughout the whole extent of the magnet; the differentials of the components A, B, C are null, and the equation (6) gives

$$\rho = 0;$$

there is magnetism therefore on the surface only.

The potential reduces then to

$$V = I \int \frac{dS \cos\theta}{r},$$

or, if dS_1 is the projection of dS, on a plane perpendicular to the magnetisation,

$$V = I \int \frac{dS_1}{r}.$$

All the elements of volume being magnetised parallel to each other, the magnetic moment of the whole, is equal to the sum of the moments of all the elementary volumes; we have, then,

$$\varpi = \int I dv = I \int dv = vI.$$

Thus, *the magnetic moment of a uniform magnet is equal to the product of the volume by the strength of the magnetisation.*

The expression for the surface density

$$\sigma = I \cos\theta$$

shows that the external action of a body magnetised uniformly is equivalent to that of two layers of gliding (157)—that is to say, of two layers which would result from the superposition of two homogeneous magnetic masses of densities $\pm\rho$ equal and of opposite signs, of which the positive part has been displaced parallel to the magnetisation by a quantity δ such that $\rho\delta = I$.

We have said (308) that we may explain the action of the earth by an infinitely small magnet placed at the centre, or by two layers of gliding; we see that we may also suppose the earth uniformly magnetised; this latter condition being equivalent to the two others.

321. Force in the Interior of a Magnet.—We cannot determine the magnetic action in the body of a magnet itself without making a cavity in which a small test magnet may be placed; but the creation of a free surface in the interior of a magnet is equivalent to the formation of a surface layer having in general a finite action on the points which it contains, and this action depends on the form of the cavity.

A mass placed in the cavity is then outside the acting masses, and the force which it undergoes may be determined in the usual manner. This force is the resultant of two others, one due to external masses and the other to the surface layer of the cavity; the second force depends on the shape and orientation of the cavity while the former is independent of it.

The strength of magnetisation may, moreover, be regarded as constant in magnitude and in direction, throughout the whole extent of the infinitely small magnet which is removed; it would therefore be possible to determine the second force for certain simple forms of the cavity.

322. Let us consider, in the first place, a cylindrical cavity the generating lines of which are parallel, and the bases at right angles, to the strength of magnetisation. The density of the fictive layer will be null on the lateral walls, since the perpendicular component of the magnetisation is zero at every point; on the two bases which are perpendicular to the magnetisation, the density will be uniform, equal to + I on one and − I on the other. If the extent of the base is equal to a, there will be equal and contrary magnetic masses + aI and − aI at the two ends of the cylinder.

Let us imagine the cylinder to be circular: let r be the radius of the base and $2h$ the height; the action of two layers on a point in the middle of the axis is double that of a homogeneous disc on a point of the perpendicular raised at its centre. In order to calculate this action, let us suppose the density equal to unity and the disc divided into concentric elementary zones; the component along the axis of the action of one of these zones at a distance ρ from the point in question is

$$df = \frac{2\pi r dr}{\rho^2} \cdot \frac{h}{\rho},$$

or, taking into account the relation $\rho^2 = r^2 + h^2$,

$$df = 2\pi h \frac{d\rho}{\rho^2}.$$

Integrating between the limits $\rho = h$ and $\rho = \sqrt{r^2 + h^2}$, we get

$$f = 2\pi \left(1 - \frac{h}{\sqrt{r^2 + h^2}} \right);$$

we shall have therefore, for the action of the two layers,

$$R = 2fI = 4\pi I \left(1 - \frac{h}{\sqrt{r^2 + h^2}} \right).$$

Two cases are particularly interesting—those in which one of the two qualities h and r is very great in comparison with the other. When the cylinder is very long, the ratio $\frac{r}{h}$ is very small, and the value of R tends to zero. The real force which then acts in the cavity is reduced to the action of external masses. An infinitely thin cylinder with any base will clearly give the same result; this will also be the case if we imagine in the magnet a section made by any surface parallel at each point to the lines of magnetisation, and if we suppose an infinitely thin interval between the two separated parts.

When the cylinder is reduced to a flat disc, the ratio $\frac{h}{r}$ is very small, and in the limit the value of R becomes

$$R = 4\pi I.$$

The components of this force parallel to the axes are

$$R_x = 4\pi A,$$
$$R_y = 4\pi B,$$
$$R_z = 4\pi C.$$

These values will agree also with the case of an infinitely thin section made in the magnet perpendicular to the lines of magnetisation.

If the cavity is in the form of a sphere, the sides will be covered with two equal and contrary layers distributed like layers of gliding; the action R of this layer on any point in the interior is constant, parallel to the magnetisation, and is expressed by

$$R = \frac{4}{3}\pi\sigma_0 = \frac{4}{3}\pi I.$$

For a narrow slit, the perpendicular of which makes an angle θ with the direction of magnetisation, the internal force of the surface layers is perpendicular to the slit, and has the value

$$4\pi\sigma = 4\pi I \cos\theta.$$

The influence of the shape of the cavity is well marked in these various examples.

323. MAGNETIC FORCE.—In the case of an elongated cylindrical cavity, or of a slit parallel to the lines of magnetisation, the force at a point only depends on the potential V of masses external to the volume-element which has been removed. The components of this force are

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z}.$$

The expression *magnetic force*, or *resultant force at a point of the magnetised mass*, is more especially assigned to the force thus defined.

324. MAGNETIC INDUCTION.—If the cavity is a very flattened cylinder, or an infinitely thin slit perpendicular to the lines of magnetisation, the components of the true force F_1 have the values

$$\begin{aligned} X_1 &= X + R_x = X + 4\pi A, \\ Y_1 &= Y + R_y = Y + 4\pi B, \\ Z_1 &= Z + R_z = Z + 4\pi C. \end{aligned} \tag{7}$$

The force F_1 plays an important part in the study of magnetisation by influence; it is called *magnetic induction.*

The sum of the three partial differentials of the function F_1 gives the equation

$$\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y} + \frac{\partial Z_1}{\partial z} = -\Delta V + 4\pi\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)$$

$$= 4\pi\rho - 4\pi\rho = 0.$$

Hence magnetic induction satisfies Laplace's equations both for points inside and outside the magnetised media. It is, moreover, identical with the magnetic force for all external points, since the magnetisation I, and its components A, B, and C, are then equal to zero. Magnetic induction has therefore the same properties as electrostatic induction (116).

A *line of induction* is a line to which the force of induction is tangential at every point; a *tube of induction* is a channel bounded laterally by lines of induction; lastly, *flow of induction* across an element of surface is the product of the surface of the element by the perpendicular component of induction. Since induction satisfies Laplace's equation for all internal and external points, it follows that the flow of induction is a constant quantity throughout the whole extent of a tube of induction.

325. Different Kinds of Magnets.—We may divide magnets into distinct categories, according to the manner in which the intensity of magnetisation varies.

326. Magnetic Solenoids.—A *simple solenoid* is a magnet in the form of a filament with an infinitely small constant section, at each point of which the intensity of magnetisation is itself constant and tangential to the direction of the filament.

The magnetic density is zero throughout the whole mass of the filament and on its lateral surface (310); at the ends only are two equal and opposite magnetic masses; if I is the strength of magnetisation and a the section of the filament, the absolute value of these two masses is

$$m = a\mathrm{I}.$$

This product $a\mathrm{I}$ may be called the *magnetic power* of the solenoid.

If, while the section of the filament, and the intensity of the magnetisation are variable, the product $a\mathrm{I}$ remains constant, the system will still form a simple magnetic solenoid.

A simple solenoid acts on all external points as would a magnet whose poles were exactly at the ends.

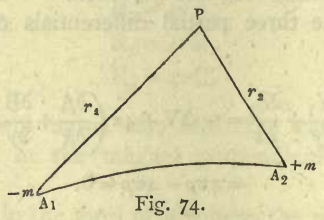

Fig. 74.

The potential at a point P (Fig. 74), at a distance r_2 from the positive pole A_2, and at a distance r_1 from the magnetic pole A_1, is thus expressed,

$$\mathrm{V} = m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = a\mathrm{I}\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

If such a solenoid is closed, the potential is everywhere zero on the outside; the force is therefore zero, and we can only discover the magnetism in the system by breaking it at a point and separating the ends.

327. A magnetic filament with a constant or variable section tangential at every point to the direction of magnetisation, and in which the magnetic power is not constant, constitutes a *complex* solenoid. Such a system may be regarded as several simple solenoids of unequal lengths united to form one bundle.

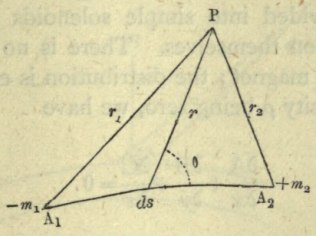

Fig. 75.

The potential P at an external point of an element of length ds (Fig. 75), whose magnetic power $m = a\mathrm{I}$ is

$$dV = \frac{m ds}{r^2}\cos\theta = -m\frac{dr}{r^2} = -a\mathrm{I}\frac{dr}{r^2}.$$

The potential at P of the whole filament is

$$V = \int_{A_1}^{A_2} -\frac{m dr}{r^2}.$$

Integrating the second member by parts, and calling $-m_1$ and $+m_2$ the masses of the extremities A_1 and A_2, we get

$$V = \frac{m_2}{r_2} - \frac{m_1}{r_1} - \int_{A_1}^{A_2} \frac{\mathrm{I}}{r}\frac{dm}{ds}ds.$$

The potential is the same as if the linear density at each point of

the filament were defined by the ratio

$$\lambda = -\frac{dm}{ds} = -\frac{d(a\mathrm{I})}{ds},$$

and we may write

$$\mathrm{V} = \frac{m_2}{r_2} - \frac{m_1}{r_1} + \int \frac{\lambda ds}{r}.$$

328. SOLENOIDAL MAGNETS.—A magnet is said to be *solenoidal* when it may be divided into simple solenoids terminating at the surface or closed upon themselves. There is no free magnetisation in the interior of the magnet; the distribution is entirely superficial.

The volume density ρ being zero, we have

$$\frac{\partial \mathrm{A}}{\partial x} + \frac{\partial \mathrm{B}}{\partial y} + \frac{\partial \mathrm{C}}{\partial z} = 0. \tag{8}$$

Conversely, if the condition (8) is satisfied, the density is zero at any point in the interior, and the magnet is solenoidal.

329. MAGNETIC SHELLS.—A simple *magnetic shell* is a magnet formed of two infinitely near equidistant surfaces, charged with equal and opposite uniform magnetic layers; or is a magnet formed of two infinitely near but not equidistant layers, always equal and of opposite signs, and such that the density at each point is inversely as their distance.

If h be the thickness of the shell at a point, and σ the density of the layer, the product $h\sigma$ must be constant; it is called the *magnetic power* of the shell.

We may also define a simple magnetic shell as being an infinitely thin plate, the magnetisation of which is perpendicular at every point to the surface, and its intensity inversely proportional to the thickness.

If Φ be the magnetic power of the shell we have

$$\Phi = h\sigma = h\mathrm{I}.$$

That portion of the shell which corresponds to an element $d\mathrm{S}$, may be regarded as an infinitely small magnet, the moment of which is

$$h\sigma d\mathrm{S} = \Phi d\mathrm{S}.$$

The potential at the point P (Fig. 76) of this element of the shell is expressed by

$$dV = \frac{h\sigma dS}{r^2}\cos\theta = \frac{\Phi dS}{r^2}\cos\theta,$$

θ being the angle formed by the perpendicular N drawn externally to the positive surface, with the right line r, which joins the point P to the element dS.

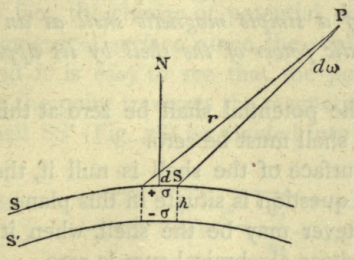

Fig. 76.

The solid angle $d\omega$, under which the element dS is seen from the point P, is given by the equation

$$dS\cos\theta = r^2 d\omega, \qquad \text{or} \qquad d\omega = \frac{dS\cos\theta}{r^2}.$$

From this it follows that

$$dV = \Phi d\omega.$$

As the factor Φ is constant, the potential of the shell at P is expressed by

(9) $$V = \Phi\omega.$$

It is important to define with precision the significance of the solid angle ω. The potential dV is positive or negative according as the point P views the positive or the negative surface of the element dS of the shell—that is to say, according as the angle θ is acute or obtuse. The angle $d\omega$, considered itself as positive or negative in

the same conditions, is the surface cut on a sphere of radius equal to unity, whose centre is the point P, by a cone described with vertex at P on the surface of this element as base; it is the *apparent surface* of this element. The angle ω, or the apparent surface of the whole shell, is therefore defined by a limited cone on the contour of this shell; it is positive or negative according as the surface of the shell which the point P views throughout the contour is itself positive or negative. The potential of the shell is therefore independent of its form, and only depends on its magnetic power and on its contour. From this follows the important theorem of Gauss:

The potential of a simple magnetic sheil at an external point is equal to the magnetic power of the shell by its apparent surface seen from this point.

In order that the potential shall be zero at this point, the apparent surface of the shell must be zero.

The apparent surface of the shell is null if, the contour being a plane, the point in question is situate in this plane.

It is zero, whatever may be the shell, when it consists of parts of opposite signs whose algebraical sum is zero.

As a particular case, if the shell forms a closed surface, the potential for any external point is zero. For a point inside the shell, the angle ω is equal to 4π, the potential is therefore constant and equal to $4\pi\Phi$; it is of the same sign as the internal surface. The value of this potential being constant both inside and outside, the action of the closed shell on any given point is zero.

330.—If two equally strong magnetic shells S and S′ (Fig. 77) have the same contour, and if their surfaces which face each other are of opposite signs, their potentials are equal for all points outside the space which they comprise; these potentials differ, on the contrary, by $4\pi\Phi$ for all points between the two surfaces. For the potential of one of the shells S is positive and equal to $\Phi\omega$, that of the other shell S′ is $-\Phi\omega'$; the difference is therefore

$$\Phi(\omega + \omega') = 4\pi\Phi.$$

In like manner, for two infinitely near points situate on each side of a magnetic shell at a finite distance from the contour, the difference of potentials is equal to $4\pi\Phi$, for it is $\Phi\omega$ for the one and $-\Phi(4\pi - \omega)$ for the other. Hence, when the point in question traverses a shell in the direction of the magnetisation—that is to say, from the negative to the positive face—the potential suddenly increases by $4\pi\Phi$.

If, while the point was fixed, the shell altered its shape so as to pass from the position S′ to the position S (Fig. 77), the potential at P would undergo the same increase of $4\pi\Phi$.

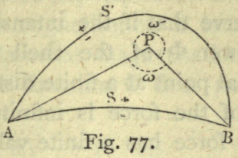

Fig. 77.

As a matter of fact, the change of potential does not take place suddenly on a geometrical surface, since the shell has necessarily a finite thickness, and it is easy to see that the potential at P varies continuously while the point traverses the magnetised layer.

For, let the shell SS′ (Fig. 78) be divided into two parallel layers

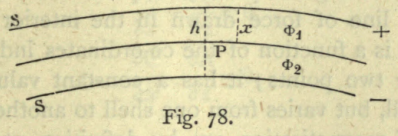

Fig. 78.

of thicknesses x and $h-x$, and of power Φ_1 and Φ_2, and consider the point P at the surface of separation of these two shells. The value of the potential at P is

$$\begin{aligned} V &= \omega\Phi_2 - (4\pi - \omega)\Phi_1 \\ &= \omega(\Phi_1 + \Phi_2) - 4\pi\Phi_1 = \omega\Phi - 4\pi\Phi_1. \end{aligned}$$

We have further

$$\Phi_1 = \sigma x = \Phi\frac{x}{h},$$

and, consequently,

$$V = \omega\Phi - 4\pi\Phi\frac{x}{h} = \Phi\left(\omega - 4\pi\frac{x}{h}\right).$$

The perpendicular action of the shell at the point P is

$$-\frac{\partial V}{\partial x} = -\Phi\frac{\partial\omega}{\partial x} + 4\pi\Phi\frac{I}{h} = -\Phi\frac{\partial\omega}{\partial x} + 4\pi I.$$

This expression, as might have been foreseen, is the perpendicular component of the induction at the point P. For we have

implicitly assumed that we placed the point P in an infinitely thin slit perpendicular to the lines of magnetisation: the term $4\pi I$ is the force which must be added to the external actions in order to have the value of the true force in the interior of the cavity.

We may further observe that if the intensity of magnetisation is finite, the magnetic power Φ of the shell is an infinitely small quantity; for any external point at a finite distance from the contour of the shell the value of the force is infinitely small, while in the interior of the shell the force has a finite value $4\pi I$, directed along the perpendicular and in an opposite direction to that of the magnetisation.

331. LAMELLAR MAGNETS.—A magnet is said to be *lamellar* when it may be divided into simple closed magnetic shells or into open shells with their edges on the surface of the magnet.

Let Φ be the sum of the magnetic powers of the shells which we meet in going from a given point to a point whose co-ordinates are x, y, z, along a line of force drawn in the interior of the magnet. This quantity Φ is a function of the co-ordinates independent of the line joining the two points; it has a constant value on the whole surface of a shell, but varies from one shell to another.

The lines of magnetisation are, by definition, at right angles to the surfaces of the elementary shells, and the strength of the magnetisation at each point is inversely as the perpendicular distance dn of two consecutive shells. We have then

$$I = \frac{d\Phi}{dn}. \tag{10}$$

332. POTENTIAL OF MAGNETISATION. — The function Φ has therefore, taking into account the sign, the same properties in reference to magnetisation as the potential in reference to external forces. Hence, by analogy, we may call the function $-\Phi$ the *potential of magnetisation.* The components of magnetisation along the axes of the co-ordinates, are respectively equal to the corresponding partial differentials of the function Φ:

$$A = \frac{\partial\Phi}{\partial x}, \qquad B = \frac{\partial\Phi}{\partial y}, \qquad C = \frac{\partial\Phi}{\partial z}.$$

From this we deduce

$$A\,dx + B\,dy + C\,dz = d\Phi. \tag{11}$$

The first member of this equation is thus an exact differential. Conversely, if the expression $Adx+Bdy+Cdz$ is the exact differential of a function of the co-ordinates, the components of magnetisation are respectively equal to the partial differentials of this function, and the magnetisation is lamellar.

The condition of lamellar magnetisation may be expressed by equations in which the function Φ does not appear.

We have, in fact,

$$\frac{\partial A}{\partial y}=\frac{\partial^2\Phi}{\partial x\partial y}=\frac{\partial B}{\partial x}, \text{ etc.};$$

which gives the three equations

$$\begin{aligned}&\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}=0,\\ &\frac{\partial B}{\partial z}-\frac{\partial C}{\partial y}=0,\\ &\frac{\partial C}{\partial x}-\frac{\partial A}{\partial z}=0.\end{aligned} \qquad (12)$$

333. A magnetic shell is said to be *complex* when, the magnetisation being always perpendicular at each point, the magnetic strength is not constant throughout the whole extent of the shell.

The potential at the point P of the element dS of the shell is still

$$dV=\Phi d\omega=\frac{\Phi\cos\theta}{r^2}dS,$$

and the potential of the entire shell

$$V=\int\Phi d\omega=\int\frac{\Phi\cos\theta}{r^2}dS,$$

the integral being extended to the whole surface of the shell.

When a magnet can be divided into complex magnetic shells, the strength of magnetisation is no longer inversely as the distance of two infinitely near shells, but the lines of magnetisation are still orthogonal to the surfaces of these shells, which gives the condition

$$\frac{A}{\frac{\partial\Phi}{\partial x}}=\frac{B}{\frac{\partial\Phi}{\partial y}}=\frac{C}{\frac{\partial\Phi}{\partial z}}. \qquad (13)$$

In this case, the expression $A dx + B dy + C dz$ is no longer an exact differential. We may again eliminate the function Φ between these equations, and we get

$$(14) \quad A\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) + B\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) + C\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = 0.$$

This is the condition which must be satisfied to have a complex lamellar magnetisation.

Conversely, if equation (14) is satisfied, the magnet is formed of complex magnetic shells, for the lines of magnetisation are orthogonal to a system of surfaces; unless each of the expressions in the parenthesis is separately zero, in which case the magnetisation would be lamellar, from equations (12).

334. POTENTIAL OF A SOLENOIDAL MAGNET.—The general value of the potential of a magnet is

$$V = \int \frac{\sigma}{r} dS + \int \frac{\rho}{r} dv.$$

If the magnet is solenoidal, the density ρ is everywhere zero, and the potential is reduced to

$$V = \int \frac{\sigma}{r} dS = \int \frac{I \cos \theta}{r} dS.$$

The potential of a solenoidal magnet at any internal or external point only depends then on the surface density, or on the perpendicular component of the strength of magnetisation at every point of the surface. This potential is independent of the manner in which the internal magnetisation varies, or in other words, on the internal form of the solenoidal filaments which terminate at the surface, as well as of the existence of closed filaments.

We may suppose, for instance, that the magnetism of the earth is produced by solenoidal filaments, maintained in the surface rocks at a low temperature, and terminating on the surface in such a way as to produce a distribution equivalent to that of a uniform magnetisation.

335. POTENTIAL OF A LAMELLAR MAGNET.—If the magnet is lamellar, it consists of closed magnetic shells, and of open shells with their contour on the surface. The force outside only depends then

on the form and position of the edge of the open shells—that is to say, of the infinitely thin zones cut on the surface by two adjacent shells, and not at all on the form of the shells.

For a point in the interior, the force in a slit between two shells, or the magnetic induction, will be obtained by combining the action determined by these successive zones, with a force in the opposite direction to the magnetisation at the point in question, and equal to $4\pi I$. The potentials by means of which these forces may be expressed, are directly obtained from the following considerations.

Let us first of all disregard the closed shells, and suppose that after having removed all the open shells which the magnet contains, we replace them by shells respectively of the same power terminated by the same edge, but applied on the surface itself; this operation would be realised physically if each of the shells were formed of

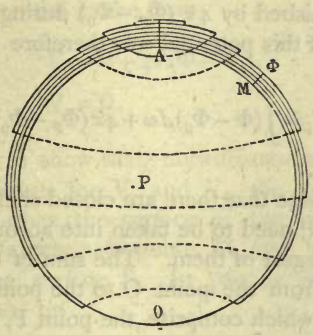

Fig. 79.

an elastic membrane, fixed by its edge, which could be stretched so as to be applied on the surface of the magnet without modifying its magnetic strength. Let us assume, for instance, that in Fig. 79 all these shells have their positive faces turned upwards, and that they are made to cover the point A of the surface of the magnet where the function Φ has its maximum value.

The entire surface will then be occupied by a series of shells, the superposition of which forms a complex shell, and produces at every point outside, the same potential as the magnet itself.

Let us now consider a point P in the interior. The potential has not changed by the fact of the transformation of those shells which passed between A and P; but for each of the other shells which have been traversed by the point P, the potential is less by $4\pi d\Phi$. Let then $-\Phi_p$ be the potential of magnetisation at P, and

$-\Phi_0$ the value of this potential at the point O of the surface where the function Φ is a minimum; during the transformation the potential at P will have diminished by the product of 4π by the sum of the magnetic powers of all the shells between the points P and O—that is by $4\pi(\Phi_p - \Phi_0)$, and this quantity must be added to the new potential at the point P to give it the value which it originally had.

At any point M of the resultant superficial shell thus formed, the magnetic strength is equal to the sum $\int d\Phi$ of that of the shells which have been superposed there; it is therefore equal to $\Phi - \Phi_0$, calling $-\Phi$ the value at this point of the original potential of magnetisation. Consequently, the potential of all the layers on the point P is equal to $\int (\Phi - \Phi_0)\, d\omega$.

If the point P is not surrounded by closed shells, the potential at this point has diminished by $4\pi\ (\Phi_p - \Phi_0)$ during the transformation; the original value of this potential was therefore

$$(15) \qquad V_i = \int (\Phi - \Phi_0) d\omega + 4\pi(\Phi_p - \Phi_0).$$

Let us now suppose that there are closed shells; only those which comprise the point P need to be taken into account. Let Φ_1 be the value of Φ on the largest of them. The sum of the magnetic powers of the open shells from the point O to the point P is $\Phi_1 - \Phi_0$; that of the closed shells which comprise the point P, and which have not been displaced by the preceding transformation, is equal to $\Phi_p - \Phi_1$.

The potential at the point P is then

$$V_i = \int (\Phi - \Phi_0) d\omega + 4\pi(\Phi_1 - \Phi_0) + 4\pi(\Phi_p - \Phi_1),$$

or

$$V_i = \int (\Phi - \Phi_0) d\omega + 4\pi(\Phi_p - \Phi_0).$$

It will be seen that the closed shells do not modify the expression of the internal potential.

The external potential is not changed by the transfer of the shells to the surface; it is expressed by

$$(16) \qquad V_e = \int (\Phi - \Phi_0) d\omega.$$

The two formulæ (15) and (16) may be simplified if we observe that the integral $\int d\omega$ is equal to zero for external points, and to -4π for internal points.

We get then

$$(15)' \qquad V_i = \int \Phi d\omega + 4\pi\Phi_p,$$

$$(16)' \qquad V_e = \int \Phi d\omega.$$

Denoting by Ω a function defined by the ratio

$$(17) \qquad \int (\Phi - \Phi_0) d\omega = \Omega,$$

we might put the potential in the form

$$(15)'' \qquad V_i = \Omega + 4\pi(\Phi - \Phi_0).$$

$$(16)'' \qquad V_e = \Omega.$$

336. It is easy to show that, notwithstanding the difference in form of the expressions for V_e and V_i, the potential varies in a continuous manner when the surface of the magnet is traversed. For consider two infinitely near points M_e and M_i, one without and the other within the surface S. In passing from M_e to M_i the function Ω diminishes by

$$4\pi \int d\Phi = 4\pi(\Phi - \Phi_0).$$

Hence, on both sides of the surface, we have

$$(18) \qquad \Omega_e = \Omega_i + 4\pi(\Phi - \Phi_0).$$

The magnetic potentials at M_e and M_i are

$$V_e = \int (\Phi - \Phi_0) d\omega = \Omega_e,$$

$$V_i = \int (\Phi - \Phi_0) d\omega + 4\pi(\Phi - \Phi_0),$$

$$= \Omega_i + 4\pi(\Phi - \Phi_0) = \Omega_e;$$

the two values are therefore equal.

337. POTENTIAL OF INDUCTION.—The function Ω plays, in reference to the induction F_1, the same part as the function V in reference to the magnetic resultant F. For the values of the components of the force F_1 are (326)

$$X_1 = -\frac{\partial V}{\partial x} + 4\pi A,$$

$$Y_1 = -\frac{\partial V}{\partial y} + 4\pi B,$$

$$Z_1 = -\frac{\partial V}{\partial z} + 4\pi C.$$

We know, on the other hand, that we have (332)

$$A = \frac{\partial \Phi}{\partial x}, \qquad B = \frac{\partial \Phi}{\partial y}, \qquad C = \frac{\partial \Phi}{\partial z}.$$

From the equation

$$V = \Omega + 4\pi(\Phi - \Phi_0)$$

we deduce

$$-\frac{\partial V}{\partial x} = -\frac{\partial \Omega}{\partial x} - 4\pi\frac{\partial \Phi}{\partial x} = -\frac{\partial \Omega}{\partial x} - 4\pi A,$$

$$-\frac{\partial V}{\partial y} = -\frac{\partial \Omega}{\partial y} - 4\pi\frac{\partial \Phi}{\partial y} = -\frac{\partial \Omega}{\partial y} - 4\pi B,$$

$$-\frac{\partial V}{\partial z} = -\frac{\partial \Omega}{\partial z} - 4\pi\frac{\partial \Phi}{\partial z} = -\frac{\partial \Omega}{\partial z} - 4\pi C,$$

and, therefore,

$$X_1 = -\frac{\partial \Omega}{\partial x},$$

(19)

$$Y_1 = -\frac{\partial \Omega}{\partial y},$$

$$Z_1 = -\frac{\partial \Omega}{\partial z}.$$

The components of the induction F_1 are therefore equal and of opposite sign to the partial differentials of the function Ω.

On the other hand, the functions V and Ω are identical for all points external to the magnetised media—points for which the induction and the magnetic force are themselves identical. Hence the function

$$\Omega = \int (\Phi - \Phi_0) d\omega$$

may be considered as the *potential of magnetic induction* of a lamellar magnet.

338. POTENTIAL ENERGY OF MAGNETS.—The general expression for the energy of a permanent magnet in a magnetic field produced by an invariable system, where m is the magnetic mass situate at the point where the potential of the field is V, is

$$W = \sum mV,$$

or again, as a function of the surface density and of the volume density of the magnetism,

$$W = \int V\sigma dS + \int V\rho dv.$$

This energy is the work which must be expended to bring the magnet in question from an infinite distance to the position which it occupies, or conversely the work done in moving it to an infinite distance.

In order to express the energy as a function of the intensity of magnetisation, we must replace the densities by their known values; but it is simpler to consider the problem directly. A volume element dv, the magnetic moment of which is $I dv$, is equivalent to a small magnet of mass m, and length ds, parallel to the direction of magnetisation. If V and V′ are the potentials of the field at the points at which are the masses $-m$ and $+m$, the energy of this element of volume is

$$dW = m(V' - V) = mds \frac{V' - V}{ds} = I dv \frac{\partial V}{\partial s}.$$

If δ be the angle which the direction of magnetisation makes with the direction of the field, and dn the perpendicular distances of the two equipotential surfaces V and V′ at the point in question, we have

$$\frac{\partial V}{\partial s} = \frac{dV}{dn} \cos\delta = -F \cos\delta = -(X\lambda + Y\mu + Z\nu),$$

X, Y and Z being the components of the force of the field, λ, μ and ν the direction cosines of the directions of magnetisation. The expression for the elementary energy is therefore

$$dW = -(XI\lambda + YI\mu + ZI\nu)\,dv = -(AX + BY + CZ)\,dv,$$

and hence the energy of the whole magnet is

$$W = -\int (AX + BY + CZ)\,dv. \tag{20}$$

If the field is uniform, the components X, Y and Z are constant. If α, β and γ are the cosines of the angles of the force F with the axes, we get

$$W = -F\left[\alpha\int A\,dv + \beta\int B\,dv + \gamma\int C\,dv\right].$$

If K be the magnetic moment of the magnet, l, m and n the cosines of the angles which the magnetic axis makes with the axes of the co-ordinates, we have

$$\int A\,dv = Kl, \qquad \int B\,dv = Km, \qquad \int C\,dv = Kn,$$

and the energy becomes

$$W = -FK(\alpha l + \beta m + \gamma n) = -FK\cos\delta, \tag{21}$$

δ being the angle which the magnetic axis makes with the direction of the field. This result may be written directly.

The energy is a minimum and equal to $-FK$, and therefore the equilibrium is stable when the angle δ is zero—that is to say, when the magnetic axis is parallel to the direction of the field. The equilibrium is unstable if these two directions are opposite; the energy is then a maximum and equal to FK. The energy, lastly, is zero if the two directions are at right angles.

339. ENERGY OF A MAGNETIC SHELL.—If the system is a simple magnetic shell S, the magnetic moment of a surface-element of the shell is ΦdS and the value of its potential energy in the field is

$$dW = -\Phi(\lambda X + \mu Y + \nu Z)\,dS;$$

the energy of the shell is therefore

$$W = -\Phi\int (\lambda X + \mu Y + \nu Z)\,dS.$$

The quantity in brackets $(X\lambda + Y\mu + Z\nu)$ represents the projection F_n of the force of the field on the perpendicular to the shell; the product $F_n dS$ is the flow of force of the field corresponding to the element dS; this flow is positive when it traverses the shell from the negative to the positive face, and negative when in the opposite direction. Hence the integral of the second member simply expresses the value of the flow limited to the edge, and therefore is independent of the form of the surface to which it is attached. Let Q be the value of this flow, the expression for the potential energy of the shell is

$$W = -\Phi Q. \tag{22}$$

Consequently, *the potential energy of a shell is equal to the product, with the contrary sign, of the power of the shell by the flow of force which penetrates its negative surface.*

340. This result may be directly obtained. For the energy of a mass m, in the field of a simple magnetic shell, is expressed by

$$dW = mV = m\Phi\omega = \Phi m\omega.$$

But the product $m\omega$ is the flow of force which starts from the point in the angle ω, and which therefore traverses the shell entering by the positive surface. The flow dQ which enters by the negative surface has the same value with the contrary sign $-m\omega$. We have thus

$$dW = -\Phi dQ.$$

But the energy of a magnetic system in the field of the shell is the sum of the energies of different masses; it is therefore the product, taken with the opposite sign, of the magnetic strength Φ of the shell by the sum of the flows of force which traverse it—that is to say, by the flow of force which starts from the system and enters the shell by the negative face.

341. If this system is a second shell S′, the flow of force Q is proportional to the magnetic strength Φ' of this second shell, and we may write $Q = M\Phi'$, the coefficient M being the flow of force which the former shell would receive, if the power of the second were equal to unity. The energy of the first shell, in the field of the second, is therefore

$$W = -\Phi\Phi' M. \tag{23}$$

The energy of the second shell in the field of the first has the same value, and is expressed in the same way,

$$W = -\Phi'\Phi M',$$

as a function of the flow of force which proceeding from the first would traverse the second; from it we infer

$$M = M'. \tag{24}$$

Thus *when two magnetic shells of equal strengths are in presence of each other, the flow of force which starts from one and traverses the other, entering by the negative face, is the same for both.*

It will be observed how analogous this property is with the theorem demonstrated above (63) relative to the electrostatic induction between two conductors.

342. Equation (22) shows that the energy of a shell in a magnetic field only depends on the flow of force which crosses the surface bounded by the contour of the shell, and that it is independent of the form of this surface. This energy, and therefore the force exerted on the shell, may be expressed then as a function of the curve of the edge.

In like manner the reciprocal energy of two shells given by equation (23) only depends on the two edges; this energy and the reciprocal force may then be expressed as a function of the two curves which bound the shells.

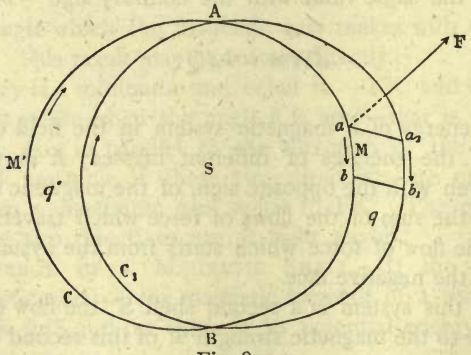

Fig. 80.

343. Action of a Field on a Shell.—Consider a shell S (Fig. 80) placed in any given magnetic field. When the shell experiences an infinitely small displacement, the increase of the potential energy is

$$dW = -\Phi dQ,$$

dQ being the increase of the flow of force which traverses the negative face of the shell. The work dT of the magnetic forces being equal and of opposite sign to dW, we have

$$dT = \Phi dQ.$$

As the form of the shell is a matter of indifference, we may suppose that it forms part of a continuous surface S, passing through the positions C and C_1 which the edge occupies, and that this only makes it glide on the surface.

The work of the magnetic forces is proportional to the excess of the flow of force which traverses the surface bounded by the edge C_1 over that which traverses the surface bounded by the edge C. The flow of force relative to the portion common to the two shells disappears by difference, so that calling q and q' the flows which traverse the spindles AMB and AM'B, we have

$$dT = \Phi(q - q').$$

Let ab be an element of the first contour, a_1b_1 its new position after the displacement, F the force of the field at this point. To obtain the part dq of the flow relative to the displacement, we must multiply the force F, by the projection of the parallelogram abb_1a_1 which this element has described, on a plane perpendicular to this force.

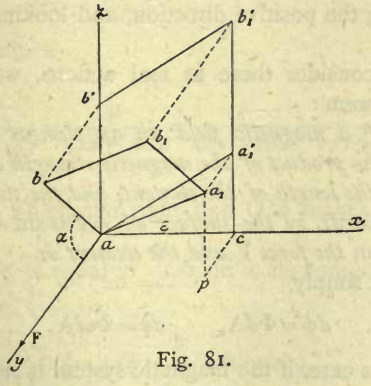

Fig. 81.

Finally, in order the better to see the geometrical signification of this product, let us imagine an observer laying along the curve C so that, looking at the shell, he has the negative face on his right hand. The positive direction of the arcs is that of a moving body which goes from the feet to the head of the observer. Let us take

the plane Fds for that of yz, and the direction of the force F as axis of y (Fig. 81). Let α be the angle which the element ds, calculated in the positive direction, makes with the force F. The projection of the parallelogram abb_1a_1 on the plane of the xz perpendicular to the force F is a new parallelogram $ab'b'_1a'_1$. We may consider this latter as having for base $ab' = ds \sin \alpha$, and for height ac—that is to say, the abscissa of the point a_1, or the projection ϵ of the displacement aa_1 on the axis ax perpendicular to the plane Fds. We have then

$$dq = \mathrm{F}ds \sin \alpha \times \epsilon. \tag{25}$$

The corresponding work is

$$\Phi dq = \Phi \mathrm{F}ds \sin \alpha \times \epsilon.$$

This work is the same as if the element ds were subjected to the action of a force

$$d\phi = \Phi \mathrm{F}ds \sin \alpha$$

parallel to the axis of x—that is, perpendicular to the plane Fds. We are thus led to this important theorem:

The action of a magnetic field on a shell is equivalent to that of a system of forces applied at the different elements of the edge.

The force, which we must suppose applied at each element, is perpendicular to the plane which passes through the element, and to the direction of the field, and is on the left of an observer placed in the element along the positive direction, and looking at the direction of the force F.

344. If we consider these as real actions, we may enunciate the following theorem:

The action of a magnetic field on an element of the edge of a shell, is equal to the product of the magnetic strength of the shell, by the force of the field, the length of the element, and the sine of the included angle—in other words, by the surface $d\mathrm{A} = \mathrm{F}ds \sin \alpha$ *of the parallelogram constructed on the force* F *and the element ds.*

We have then simply

$$d\phi = \Phi d\mathrm{A}, \qquad dq = \Phi \epsilon d\mathrm{A}.$$

As a particular case, if the magnetic system is reduced to a single mass m at a distance r from the element ds, the force F is equal to $\frac{m}{r^2}$, and the elementary action becomes

$$d\phi = \Phi \frac{m}{r^2} ds \sin \alpha.$$

Hence, *the action of a magnetic pole on an element of the edge of a shell is proportional to the magnetic strength of the shell, to the sine of the angle which the element makes with right line joining the pole to the element, and inversely as the square of the distance.*

345. We may further remark that dq represents the flow of force cut by an element ds during the displacement aa_1, the flow of force thus cut being counted positive or negative, according as the displacement is to the left or right of the observer whose position has been defined as above. From this follows the theorem:

The work of magnetic forces during the displacement is equal to the product of the shell by the sum of the flows of force cut by each of the elements of the contour.

346. Let us suppose that the external system reduces to a magnetic mass equal to unity, and placed at the origin O of the co-ordinates. Let C be the edge of the shell, and ds an element at the distance r, at a point M whose co-ordinates are x, y, and z.

Let λ, μ, and ν be the cosines of the angles which the force $d\phi$ makes with the axes. This force being perpendicular to the element ds and to the straight line OM along which is directed the action $\frac{1}{r^2}$ proceeding from the point O, we have the ratios

$$\lambda x + \mu y + \nu z = 0,$$
$$\lambda dx + \mu dy + \nu dz = 0;$$

from which we deduce

$$\frac{\lambda}{ydz - zdy} = \frac{\mu}{zdx - xdz} = \frac{\nu}{xdy - ydx} = \frac{1}{rds \sin \alpha},$$

α being the angle which the right line OM makes with the element ds.

As the force $d\phi$ is equal to $\frac{\Phi}{r^2} ds \sin \alpha$, its components $d\xi$, $d\eta$, and $d\zeta$ are

$$d\xi = \lambda d\phi = \frac{\Phi}{r^3}(ydz - zdy),$$

$$d\eta = \mu d\phi = \frac{\Phi}{r^3}(zdx - xdz),$$

$$d\zeta = \nu d\phi = \frac{\Phi}{r^3}(xdy - ydx).$$

The action of the point O on the shell will be obtained by extending these expressions to the whole surface. Lastly, the action F of the shell on the point O, passes through this point, and the components X, Y, and Z of this force, are equal and of opposite sign to those of the action of the point on the shell; we have then

$$(26)\qquad \begin{aligned} X &= \Phi \int \frac{zdy - ydz}{r^3} = \Phi \int \frac{z^2}{r^3} d\left(\frac{y}{z}\right), \\ Y &= \Phi \int \frac{xdz - zdx}{r^3} = \Phi \int \frac{x^2}{r^3} d\left(\frac{z}{x}\right), \\ Z &= \Phi \int \frac{ydx - xdy}{r^3} = \Phi \int \frac{y^2}{r^3} d\left(\frac{x}{y}\right). \end{aligned}$$

347. Reciprocal Action of Two Shells.—We may now determine the reciprocal action of two shells S and S′. The action of S on S′ may be considered as the resultant of actions, determined by the previous rule, which the shell S would exert on each of the elements ds' of the contour C′ of the second shell.

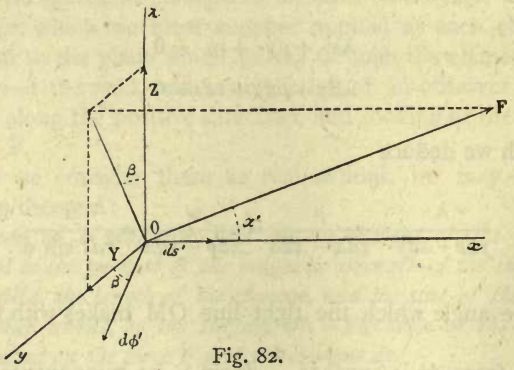

Fig. 82.

Let us suppose that one of these elements ds' is at O (Fig. 82) and is directed along the axis of x. The action $d\phi'$ which is exerted on this element is equal to $\Phi' F ds' \sin \alpha'$, and is situated in the plane of zy; the components of this force are

$$\begin{aligned} d\xi' &= 0, \\ d\eta' &= \Phi' F ds' \sin \alpha' . \cos \beta = \Phi' Z ds', \\ d\zeta' &= -\Phi' F ds' \sin \alpha' . \sin \beta = -\Phi' Y ds'; \end{aligned}$$

which, expressing the forces Z and Y as a function of the co-ordinates x, y and z of the point M where the element ds is situate, gives

$$
\begin{aligned}
d\xi' &= 0, \\
d\eta' &= \Phi\Phi' ds' \int \frac{ydx - xdy}{r^3}, \\
d\zeta' &= \Phi\Phi' ds' \int \frac{zdx - xdz}{r^3}.
\end{aligned}
\tag{27}
$$

We may also consider the action of the edge C on the element ds', as the resultant of the direct actions which each of the elements ds would exert on the element ds'. The only condition imposed on this elementary action is, that the integral of the partial components extended to the edge C shall reproduce the preceding expressions.

348. In accordance with this, the simplest solution for the action of ds on ds' is a force f, the components of which parallel to the axes f_x, f_y, f_z are, representing by a the product $\Phi\Phi' ds'$,

$$
\begin{aligned}
f_x &= 0, \\
f_y &= -a\frac{xdy - ydx}{r^3} = -a\frac{x^2}{r^3} d\left(\frac{y}{x}\right), \\
f_z &= -a\frac{xdz - zdx}{r^3} = -a\frac{x^2}{r^3} d\left(\frac{z}{x}\right).
\end{aligned}
\tag{28}
$$

349. To each of the components of the elementary action an exact differential of the co-ordinates x, y, and z may be added, since the integrals extended to the contour C will give values of zero for these terms. There is therefore an infinite number of expressions by which the actions of the elements of two magnetic shells may be expressed.

Let X, Y, and Z be functions of the co-ordinates x, y, and z; the problem will be satisfied if we take as components of the action

$$
\begin{aligned}
f_x &= a d\mathrm{X}, \\
f_y &= a\left[d\mathrm{Y} - \frac{x^2}{r^3} d\left(\frac{y}{x}\right)\right], \\
f_z &= a\left[d\mathrm{Z} - \frac{x^2}{r^3} d\left(\frac{z}{x}\right)\right].
\end{aligned}
\tag{29}
$$

350. Let us, for instance, impose the condition on this force, that it shall be directed along the right line which joins the elements, so that we have

$$\frac{f_x}{x}=\frac{f_y}{y}=\frac{f_z}{z}=\frac{f}{r};$$

it follows that

$$d\mathrm{Y}-\frac{x^2}{r^3}d\left(\frac{y}{x}\right)=\frac{y}{x}d\mathrm{X},$$

or

$$d\mathrm{Y}=\frac{y}{x}d\mathrm{X}+\frac{x^2}{r^3}d\left(\frac{y}{x}\right),$$
$$d\mathrm{Z}=\frac{z}{x}d\mathrm{X}+\frac{x^2}{r^3}d\left(\frac{z}{x}\right).$$

In order that the second members of these two latter equations shall be exact differentials of a function of the co-ordinates, we must have

$$d\mathrm{X}=d\left(\frac{x^2}{r^3}\right),$$

and, therefore,

$$d\mathrm{Y}=d\left(\frac{y}{x}\cdot\frac{x^2}{r^3}\right)=d\left(\frac{xy}{r^3}\right),$$
$$d\mathrm{Z}=d\left(\frac{z}{x}\cdot\frac{x^2}{r^3}\right)=d\left(\frac{xz}{r^3}\right).$$

The components of the elementary force will then be

$$f_x=ad\left(\frac{x^2}{r^3}\right),$$
$$f_y=a\left[d\left(\frac{xy}{r^3}\right)-\frac{x^2}{r^3}d\left(\frac{y}{x}\right)\right], \qquad (30)$$
$$f_z=a\left[d\left(\frac{xz}{r^3}\right)-\frac{x^2}{r^3}d\left(\frac{z}{x}\right)\right].$$

The force itself may be determined by the ratio

$$f=\frac{r}{x}f_x,$$

which gives

$$f=a\frac{r}{x}d\left(\frac{x^2}{r^3}\right)=\frac{2a}{r^2}\left[dx-\frac{3}{2}\frac{x}{r}dr\right]=\frac{2a}{r^2}\left[\frac{\partial x}{\partial s}-\frac{3}{2}\frac{x}{r}\frac{\partial r}{\partial s}\right]ds.$$

If θ and θ' are the angles which the elements ds and ds' make respectively with the right line OM which joins them, and ϵ the angle of these two elements, we have

$$\frac{\partial x}{\partial s} = \cos \epsilon,$$

$$\frac{x}{r} = \cos \theta',$$

$$\frac{\partial r}{\partial s} = \cos \theta,$$

and we get

$$f = \frac{2a}{r^2}\left[\cos \epsilon - \frac{3}{2}\cos \theta \cos \theta'\right] ds.$$

If we consider the action of ds upon ds' and take the distance r as positive in the direction MO, we must change the sign of the force and replace the angles θ and θ' by $\pi - \theta$ and $\pi - \theta'$, which does not change the sign of the product of the cosines.

Let us represent by $d^2\psi$ the action of ds upon ds', which is an infinitely small quantity of the second order, and consider this force to be repulsive; we shall have finally

$$(31) \qquad d^2\psi = -\frac{2\Phi\Phi' ds ds'}{r^2}\left(\cos \epsilon - \frac{3}{2}\cos \theta \cos \theta'\right).$$

351. We may give another form to this expression, which is more convenient for estimating the work.

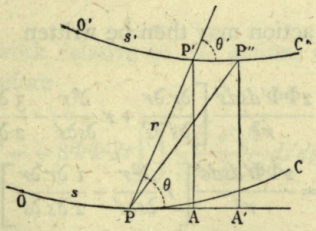

Fig. 83.

Let C and C' (Fig. 83) be the edges of two shells, ds and ds' the elements at P and P', and let us count the arcs s and s' respectively from the fixed points O and O'.

From the figure we have

$$\cos\theta = -\frac{\partial r}{\partial s},$$

$$\cos\theta' = \frac{\partial r}{\partial s'}.$$

From the extremities P′ and P″ of the elements ds' let perpendiculars P′A and P″A′ be drawn to the tangent to the curve; we get

$$PA = r\cos\theta,$$

$$PA' = r\cos\theta + \frac{\partial(r\cos\theta)}{\partial s'}ds',$$

from which we deduce

$$AA' = \frac{\partial(r\cos\theta)}{\partial s'}ds'.$$

On the other hand, the distance AA′ is the projection of the element ds' on the tangent at P to the curve s, which gives

$$AA' = \frac{\partial(r\cos\theta)}{\partial s'}ds' = ds'\cos\epsilon,$$

and, therefore,

$$\cos\epsilon = \frac{\partial(r\cos\theta)}{\partial s'} = -\frac{\partial\left(r\frac{\partial r}{\partial s}\right)}{\partial s'} = -\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} - r\frac{\partial^2 r}{\partial s\partial s'}.$$

The elementary action may then be written

$$d^2\psi = \frac{2\Phi\Phi' ds ds'}{r^2}\left[\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + r\frac{\partial^2 r}{\partial s\partial s'} - \frac{3}{2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right]$$

$$= \frac{2\Phi\Phi' ds ds'}{r^2}\left[r\frac{\partial^2 r}{\partial s\partial s'} - \frac{1}{2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right].$$

We have further

$$\frac{\partial^2\sqrt{r}}{\partial s\partial s'} = \frac{1}{2\sqrt{r}}\frac{\partial^2 r}{\partial s\partial s'} - \frac{1}{4r\sqrt{r}}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} = \frac{1}{2r\sqrt{r}}\left[r\frac{\partial^2 r}{\partial s\partial s'} - \frac{1}{2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right]$$

which gives finally

$$(32) \qquad d^2\psi = \frac{4\Phi\Phi' ds ds'}{\sqrt{r}} \frac{\partial^2 \sqrt{r}}{\partial s \partial s'}.$$

352. To determine the relative energy of the system, let us suppose that the shell S′ moves away, and that during the time dt the distance r of two elements varies by $\frac{\partial r}{\partial t} dt$ or $2\sqrt{r}\frac{\partial \sqrt{r}}{\partial t} dt$. The corresponding elementary work of the force $d^2\psi$ is equal to $d^2\psi \frac{\partial r}{\partial t} dt$, so that the total work $d^2\mathrm{T}$ relative to the element ds for the time dt is

$$d^2\mathrm{T} = 8\Phi\Phi' dt ds \int \frac{\partial \sqrt{r}}{\partial t} \frac{\partial^2 \sqrt{r}}{\partial s \partial s'} ds'.$$

Integrating by parts, we have

$$\int \frac{\partial \sqrt{r}}{\partial t} \frac{\partial^2 \sqrt{r}}{\partial s \partial s'} ds' = \left[\frac{\partial \sqrt{r}}{\partial t} \frac{\partial \sqrt{r}}{\partial s} \right] - \int \frac{\partial \sqrt{r}}{\partial s} \frac{\partial^2 \sqrt{r}}{\partial s' \partial t} ds'.$$

The first term of the second member is zero for the closed surface C′, which gives

$$d^2\mathrm{T} = -8\Phi\Phi' dt ds \int \frac{\partial \sqrt{r}}{\partial s} \frac{\partial^2 \sqrt{r}}{\partial s' \partial t} ds'.$$

The elementary work relative to the actions of the two circuits in the time dt is therefore

$$d\mathrm{T} = -8\Phi\Phi' dt \int\int \frac{\partial \sqrt{r}}{\partial s} \frac{\partial^2 \sqrt{r}}{\partial s' \partial t} ds' ds.$$

This work being symmetrical in reference to the edges C and C′, we have also

$$d\mathrm{T} = -8\Phi\Phi' dt \int\int \frac{\partial \sqrt{r}}{\partial s'} \cdot \frac{\partial^2 \sqrt{r}}{\partial s \partial t} ds ds'.$$

From which follows, taking the half sum of these expressions

$$d\mathrm{T} = -4\Phi\Phi' dt \int\int \left[\frac{\partial\sqrt{r}}{\partial s}\frac{\partial^2\sqrt{r}}{\partial s'\partial t} + \frac{\partial\sqrt{r}}{\partial s'}\frac{\partial^2\sqrt{r}}{\partial s\partial t} \right] ds ds'$$

$$= -4\Phi\Phi' dt \int\int \frac{\partial}{\partial t}\left[\frac{\partial\sqrt{r}}{\partial s}\frac{\partial\sqrt{r}}{\partial s'} \right] ds ds',$$

or

$$d\mathrm{T} = -4\Phi\Phi' d \int\int \frac{\partial\sqrt{r}}{\partial s}\frac{\partial\sqrt{r}}{\partial s'} ds ds'.$$

The relative potential energy W of the two shells is equal to the work which the forces can perform when one of the shells C′ moves to an infinite distance. We have then

$$\mathrm{W} = -4\Phi\Phi' \int\int \frac{\partial\sqrt{r}}{\partial s}\frac{\partial\sqrt{r}}{\partial s'} ds ds'. \tag{33}$$

353. This expression for the energy may be put under several different forms.

We have, in fact,

$$\frac{\partial\sqrt{r}}{\partial s}\frac{\partial\sqrt{r}}{\partial s'} = \frac{1}{4r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} = -\frac{1}{4r}\cos\theta\cos\theta' = -\frac{r}{4}\frac{\partial r}{\partial s}\frac{\partial\frac{1}{r}}{\partial s'}.$$

We may then write

$$\mathrm{W} = \Phi\Phi' \int\int \frac{\cos\theta\cos\theta'}{r} ds ds' = \Phi\Phi' \int\int r\frac{\partial r}{\partial s}\frac{\partial\frac{1}{r}}{\partial s'} ds ds'$$

$$= \Phi\Phi' \int ds \int r\frac{\partial r}{\partial s}\frac{\partial\frac{1}{r}}{\partial s'} ds'.$$

Integrating this latter expression by parts, we have

$$\int r\frac{\partial r}{\partial s}\frac{\partial\frac{1}{r}}{\partial s'} ds' = \left[\frac{\partial r}{\partial s}\right] - \int \frac{1}{r}\frac{\partial\left(r\frac{\partial r}{\partial s}\right)}{\partial s'} ds'.$$

The first term of the second member is null, being extended to the closed contour C′; we have then

$$\int r \frac{\partial r}{\partial s} \frac{\partial \frac{I}{r}}{\partial s'} ds' = - \int \frac{I}{r} \frac{\partial \left(r \frac{\partial r}{\partial s} \right)}{\partial s'} ds' = \int \frac{\cos \epsilon}{r} ds',$$

and therefore

$$W = \Phi\Phi' \int\int \frac{\cos \epsilon}{r} ds ds'. \tag{34}$$

This remarkable formula is due to F. E. Neumann. We deduce from it for the value of the coefficient M (341), which expresses the flow of force common to the two sheets, each supposed equal to unity,

$$M = - \int\int \frac{\cos \epsilon}{r} ds ds'. \tag{35}$$

CHAPTER III.

PARTICULAR CASES.

354. Potential of a Uniform Magnet.—The magnetic action of a body uniformly magnetised being equivalent to that of two layers of gliding (320), the potential V may be readily deduced from that of a homogeneous mass which would fill the volume. Let P be the value of this potential at a point M, when the density of the mass is equal to unity, its value will be ρP if the density is ρ.

The potential of the system of the two layers is evidently the sum of the potential ρP of the positive mass, and of the potential $-\rho P'$ of an identical negative mass which has been displaced in the opposite direction to that of the magnetisation, by an infinitely small quantity $dx = \delta$. The potential $\rho P'$ is that of the positive mass at the point M', whose co-ordinates are the same as that of the point M, except the abscissa parallel to the magnetisation, which has increased by dx.

We thus obtain

$$(1) \qquad V = \rho P - \rho\left(P + \frac{\partial P}{\partial x} dx\right) = -\rho\delta\frac{\partial P}{\partial x} = -I\frac{\partial P}{\partial x}.$$

Consequently the potential of a uniform magnet is equal, and of opposite sign, to the product of the intensity of magnetisation by the partial differential, referred to the direction of the magnetisation, of the potential, which a uniform mass, of density equal to unity occupying the whole volume of the body, would have at the point in question.

The components X, Y, and Z of the magnetic force are equal, and of opposite sign, to the partial differential of the potential, which gives

$$(2) \qquad X = I\frac{\partial^2 P}{\partial x^2}, \qquad Y = I\frac{\partial^2 P}{\partial x \partial y}, \qquad Z = I\frac{\partial^2 P}{\partial x \partial z}.$$

355. SPHERE.—For a sphere of volume u, for instance, the value of P at an external point at a distance r from the centre, is

$$P = \frac{u}{r};$$

from which follows

$$\frac{\partial P}{\partial x} = -\frac{u}{r^2}\frac{\partial r}{\partial x} = -\frac{ux}{r^3},$$

and therefore

$$V = Iu\frac{x}{r^3},$$

as we have previously seen (157).

In the interior of the sphere the action of a mass of unit density will be equal to $\frac{4}{3}\pi r$ (44), which, if a is the radius of the sphere, gives for the potential

$$P = \frac{2}{3}\pi(3a^2 - r^2).$$

from which we get

$$\frac{\partial P}{\partial x} = -\frac{4}{3}\pi x,$$

and consequently

$$X = -\frac{4}{3}\pi I, \qquad Y = 0, \qquad Z = 0;$$

that is to say, that the internal action of a uniformly magnetised sphere is constant, parallel, and in a contrary direction to the direction of magnetisation, a result which has already been established directly (157).

The magnetic induction in the interior of the sphere is constant and equal (324) to

$$-\frac{4}{3}\pi I + 4\pi I = \frac{8}{3}\pi I,$$

so that the total flow of induction which traverses the great circle, perpendicular to the magnetisation, is expressed by

$$\frac{8}{3}\pi \mathrm{I} \,.\, \pi a^2 = \frac{2}{3}(2\pi a)^2 \mathrm{I}.$$

356. ELLIPSOID.—Consider a homogeneous ellipsoid, the axes of which $2a$, $2b$, $2c$, are taken as axes of the co-ordinates. Denoting by L, M, N, known functions of the axes, the potential of this ellipsoid at a point in the interior, the co-ordinates of which are x, y, z, is

$$\mathrm{P} = -\frac{1}{2}(\mathrm{L}x^2 + \mathrm{M}y^2 + \mathrm{N}z^2) + \text{const.}$$

If the ellipsoid is uniformly magnetised in a direction which makes, with the axes, angles whose cosines are l, m, n, the components of the magnetisations are

$$\frac{\mathrm{A}}{l} = \frac{\mathrm{B}}{m} = \frac{\mathrm{C}}{n} = \mathrm{I},$$

and the state of the ellipsoid may be considered as being produced by the superposition of these three magnetisations, respectively parallel to the axes. The potential in the interior is

$$\mathrm{V} = -\left(\mathrm{A}\frac{\partial \mathrm{P}}{\partial x} + \mathrm{B}\frac{\partial \mathrm{P}}{\partial y} + \mathrm{C}\frac{\partial \mathrm{P}}{\partial z}\right) = -\mathrm{I}(\mathrm{L}lx + \mathrm{M}my + \mathrm{N}nz),$$

and the values of the components of the force parallel to the axes are

$$\mathrm{X} = -\mathrm{AL}, \qquad \mathrm{Y} = -\mathrm{BM}, \qquad \mathrm{Z} = -\mathrm{CN}.$$

The interior magnetic force of a uniformly magnetised ellipsoid, is therefore constant in magnitude and direction, and makes, with the axes, angles whose cosines are respectively proportional to AL, BM, and CN.

The components of the induction parallel to the axes are

$$\mathrm{X}_1 = (4\pi - \mathrm{L})\mathrm{A},$$
$$\mathrm{Y}_1 = (4\pi - \mathrm{M})\mathrm{B},$$
$$\mathrm{Z}_1 = (4\pi - \mathrm{N})\mathrm{C}.$$

Induction is therefore a constant force, which makes, with the axes, angles whose cosines are respectively proportional to $(4\pi - L)$ A, $(4\pi - M)$ B, and $(4\pi - N)$ C.

Lastly, the values of the flows of induction across the three principal sections are respectively $\pi bc\ (4\pi - L)$ A, $\pi ca\ (4\pi - M)$ B, and $\pi ab\ (4\pi - N)$ C.

357. If the magnetisation is parallel to one of the axes—the axis a, for instance—we have simply

$$F = -IL.$$

From the manner in which the layer is formed, the quantity of magnetism M_a, distributed on each of the halves of the ellipsoid, is equal to the total charge which would exist on the principal section, parallel to the two other axes, if the density were uniform and equal to I; we have then

$$M_a = \pi bcI = -\pi bc\frac{F}{L}.$$

The magnetic moment ϖ_a of the magnet thus formed, is equal to the product of the volume by the intensity, which gives

$$\varpi_a = \frac{4}{3}\pi abcI = -\frac{4}{3}\pi abc\frac{F}{L}.$$

The poles of the magnet, or the centres of gravity of each of the two layers, are at a distance a' from the centre determined by the equation

$$\varpi_a = 2a'M_a,$$

which gives

$$a' = \frac{1}{2}\frac{\varpi_a}{M_a} = \frac{2}{3}a.$$

Thus the pole of an ellipsoid uniformly magnetised in a direction parallel to one of the axes, is at a distance from the centre equal to $\frac{2}{3}$ of the length of the corresponding half-axis.

The density at a point of the surface is given by the equation

$$\sigma = I\cos\theta = I\frac{\dfrac{x}{a^2}}{\sqrt{\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}}} = I\frac{px}{a^2},$$

p being the perpendicular let fall from the centre, on the tangent plane at the point whose co-ordinates are x, y, and z.

The total charge of a zone determined by two planes perpendicular to the axis a and at a distance dx, is equal to the product of the intensity I by the difference dS of the sections of the ellipsoid corresponding to these two planes. At a distance x the section is bounded by the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1-\frac{x^2}{a^2},$$

the surface of which is

$$S = \pi bc\left(1-\frac{x^2}{a^2}\right);$$

we have then

$$dS = -\frac{2\pi bc}{a^2}x\,dx,$$

and, consequently, the charge of the zone is

$$dM_a = -\frac{2\pi bcI}{a^2}x\,dx.$$

The ratio $\dfrac{dM_a}{dx}$ of the charge of the zone to its height, which may be defined as the linear density, in reference to the axis of magnetisation, is therefore proportional to the distance of this zone from the centre of the ellipsoid.

358. When the axes of the ellipsoid are unequal, the coefficients L, M, N are given by the partial differentials of a definite elliptic integral; for the complete calculation we must refer to special treatises, and must limit ourselves to giving the results relative to ellipsoids of revolution. In this case, in fact, the problem is more simple, and the coefficients are expressed by means of the ordinary functions.

If the ellipsoid is one of revolution about the minor axis c, we have, if e is the eccentricity of the meridional ellipse,

$$a = b = \frac{c}{\sqrt{1 - e^2}},$$

$$L = M = 2\pi \left[\frac{\sqrt{1 - e^2}}{e^3 \sin e} - \frac{1 - e^2}{e^2} \right],$$

$$N = 4\pi \left[\frac{1}{e^2} - \frac{\sqrt{1 - e^2}}{e^3 \sin e} \right].$$

For an ellipsoid of revolution about the major axis,

$$b = c = a\sqrt{1 - e^2},$$

$$L = 4\pi \frac{1 - e^2}{e^2} \left[\frac{1}{2e} l. \frac{1 + e}{1 - e} - 1 \right],$$

$$M = N = 2\pi \left[\frac{1}{e^2} - \frac{1 - e^2}{2e^3} l. \frac{1 + e}{1 - e} \right].$$

Making $e = 0$ in these formulas, we find again the results already obtained for the sphere; that is to say,

$$L = M = N = \frac{4}{3}\pi,$$

$$F = -\frac{4}{3}\pi I.$$

For a very flat ellipsoid, in which the eccentricity e is near unity, we have, at the limit,

$$L = M = \pi^2 \frac{c}{a} = \pi^2 \sqrt{1 - e^2},$$

$$N = 4\pi.$$

If the ellipsoid is very elongated, we have approximately

$$M = N = 2\pi,$$

$$L = 4\pi \frac{b^2}{a^2} \left(l. \frac{2a}{b} - 1 \right),$$

and the coefficient L tends towards zero, when the eccentricity tends towards unity.

For a very flattened ellipsoid, which at the limit might be confounded with a very thin disc, the force in the interior is given by the equations

$$F = -4\pi I, \qquad \text{or} \qquad F = -\pi^2\sqrt{1-e^2}\,I,$$

according as the magnetisation is perpendicular or parallel to the plane of the disc.

For a very elongated ellipsoid we have, in like manner,

$$F = -2\pi I, \qquad \text{or} \qquad F = 4\pi^2\frac{b^2}{a^2}\left(l.\frac{2a}{b} - 1\right)I,$$

according as the magnetisation is perpendicular or parallel to the major axis.

359. CYLINDER MAGNETISED TRANSVERSELY.—The case of a cylinder might be deduced from that of an ellipsoid, but it can be easily treated directly. If we consider an unlimited circular cylinder of radius a, and density equal to unity, the mass of unit length will be $\lambda = \pi a^2$.

The external action of this cylinder, at a distance r from the axis, is equal (132) to $\frac{2\lambda}{r} = \frac{2\pi a^2}{r}$, which gives for the potential

$$P = -2\pi a^2 l.r + \text{const.}$$

If the cylinder has a uniform transverse magnetisation, and if we take the axis of x, parallel to the magnetisation, the external potential will then be

$$V = -I\frac{\partial P}{\partial x} = I\frac{2\pi a^2}{r}\frac{\partial r}{\partial x} = I\frac{2\pi a^2}{r^2}x.$$

On a point in the interior, the action of a homogeneous circular cylinder reduces to that of a cylindrical core passing through the point. This can be easily seen by reasoning analogous to that which has been applied to the sphere (42). The action of a cylinder on a point in the interior is therefore equal to $\frac{2\pi r^2}{r} = 2\pi r$, and the potential becomes

$$P = -\pi r^2 + \text{const.}$$

The potential of the uniformly magnetised cylinder is then

$$V = -I\frac{\partial P}{\partial x} = I2\pi r\frac{\partial r}{\partial x} = I2\pi x.$$

Consequently the force in the interior is constant and equal to $-2\pi I$; its direction is opposite that of the magnetisation.

The induction is also constant and has the value

$$F_1 = 4\pi I - 2\pi I = 2\pi I.$$

360. POTENTIAL OF MAGNETIC SHELLS.—We have seen that the potential of a uniform magnetic shell is equal to the product of its magnetic strength Φ by its apparent surface ω at the point in question. If the shell is not uniform the expression for the potential is

$$V = \int \Phi d\omega.$$

By a method like that of the preceding, the calculation of the potential may be reduced to the potential of a magnetic layer, so as to avoid the determination of solid angles.

The potential at a point M, of an element of the shell dS, is equal to that of two magnetic layers $\pm\sigma d$S, equal and of opposite signs, the perpendicular distance of which satisfies the condition $\sigma dn = \Phi$.

Let us denote by n the distance of the element from a fixed point on the perpendicular, on the same side as the negative face, the potential of the layer σdS may be regarded as the product of ds by a function of this distance n, so that the potential of the element of the shell will be

$$d\mathrm{S}f(n) - d\mathrm{S}f(n - dn) = d\mathrm{S}dn\frac{\partial f}{\partial n}.$$

As the distance dn of the two surfaces may be supposed to be constant, the potential of the whole shell is

$$V = \int d\mathrm{S}dn\frac{\partial f}{\partial n} = dn\int d\mathrm{S}\frac{\partial f}{\partial n}.$$

But the expression $\int f d\mathrm{S}$ represents the potential U of the positive surface of the shell; we have then

$$\mathrm{V} = dn \frac{\partial \mathrm{U}}{\partial n}. \tag{3}$$

The value of the potential Q of a layer whose density at each point is equal to the magnetic strength Φ, is $\mathrm{U}\frac{\Phi}{\sigma}$ or $\mathrm{U}dn$; from this follows

$$\mathrm{V} = \frac{\partial \mathrm{Q}}{\partial n}. \tag{4}$$

If $\frac{1}{p}$ be the distance of the point M from the element dS, the potential Q is equal to $\int p\Phi ds$, which further gives

$$\mathrm{V} = \int \Phi \frac{\partial p}{\partial n} d\mathrm{S}. \tag{5}$$

The factor p represents the potential of unit mass placed at the point M on the element dS.

361. If the shell is uniform the factor Φ is a constant. If P is the potential of a layer of density equal to unity, we shall have $\mathrm{Q} = \Phi \mathrm{P}$, and expressions (4) and (5) become

$$\mathrm{V} = \Phi \frac{\partial \mathrm{P}}{\partial n}, \tag{6}$$

$$\mathrm{V} = \Phi \int \frac{\partial p}{\partial n} d\mathrm{S}. \tag{6}'$$

362. Let us consider, for instance, a shell bounded by a plane surface; this may be replaced by a plane shell of the same strength, bounded by the same surface. Let us place this shell in the plane of yz, the positive face on the side of the x-axis. The abscissa of the point M being x, we have evidently $dx = -dn$, and therefore

$$\frac{\partial \mathrm{Q}}{\partial n} = -\frac{\partial \mathrm{Q}}{\partial x},$$

$$\mathrm{V} = -\frac{\partial \mathrm{Q}}{\partial x} = -\int \Phi \frac{\partial p}{\partial x} d\mathrm{S}. \tag{7}$$

If the shell is uniform we have

$$(7)' \qquad V = -\Phi\frac{\partial P}{\partial x} = -\Phi\int\frac{\partial p}{\partial x}dS,$$

an expression which might have been obtained by the consideration of layers of gliding (354).

363. For a shell on a sphere of radius a, the point M being at the outside, on the positive face of the shell, we have, in like manner,

$$(8) \qquad V = \frac{\partial Q}{\partial a} = \int\Phi\frac{\partial p}{\partial a}dS,$$

and, if the shell is uniform,

$$(8') \qquad V = \Phi\frac{\partial P}{\partial a} = \Phi\int\frac{\partial p}{\partial a}dS.$$

On the contrary, if the face turned away from the side of the point M is negative, we must take the expressions

$$V = -\frac{\partial Q}{\partial a} = -\int\Phi\frac{\partial p}{\partial a}dS,$$

$$V = -\Phi\frac{\partial P}{\partial a} = -\Phi\int\frac{\partial p}{\partial a}dS.$$

364. In the case of a sphere, the potential p is a homogeneous function of the degree -1 of the radius a, and of the distance r from the point M at the centre, which gives the condition

$$a\frac{\partial p}{\partial a} + r\frac{\partial p}{\partial r} = -p,$$

or

$$\frac{\partial p}{\partial a} = -\frac{1}{a}\left(p + r\frac{\partial p}{\partial r}\right) = -\frac{1}{a}\frac{\partial(pr)}{\partial r}.$$

from which follows

$$V = -\int \frac{\Phi}{a}\frac{\partial(pr)}{\partial r}dS.$$

The distances r and a being constant during the integration, we may write

$$(9) \qquad V = -\frac{1}{a}\frac{\partial}{\partial r}\int pr\Phi dS = -\frac{1}{a}\frac{\partial(rQ)}{\partial r}.$$

365. POTENTIAL OF A CIRCULAR LAYER.—The potential of a uniform shell with a circular edge may be calculated from the potential of a plane circular layer, or of any given layer—spherical, for instance—bounded by the same edge.

Consider, in the first case generally, a layer of revolution about the axis of x. For a point M the abscissa of which is x, and which is at a distance ρ from the axis, the potential P is a function of x and of ρ. If this potential be developed in increasing powers of ρ or of $\frac{1}{\rho}$, the series, from symmetry, will only contain even powers of the variable. We may therefore write

$$(10) \qquad \begin{aligned} P &= A_0 + A_2\rho^2 + A_4\rho^4 + \ldots\ldots, \\ P &= B_0 + B_2\left(\frac{1}{\rho}\right)^2 + B_4\left(\frac{1}{\rho}\right)^4 + \ldots\ldots, \end{aligned}$$

the coefficients A_0, $A_2 \ldots\ldots$, B_0, $B_2 \ldots\ldots$ being functions of x.

When x and ρ are taken as independent variables, Laplace's equation $\Delta V = 0$ becomes

$$\frac{\partial^2 P}{\partial x^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + \frac{\partial^2 P}{\partial \rho^2} = 0.$$

This condition gives for the first series, a new series developed in increasing powers of ρ, in which the coefficients of all the terms must be separately null, from which follows the general condition

$$\frac{\partial^2 A_{2n}}{\partial x^2} + (2n+2)^2 A_{2n+2} = 0.$$

We have thus successively

$$A_2 = -\frac{1}{2^2}\cdot\frac{\partial^2 A_0}{\partial x^2},$$

$$A_4 = -\frac{1}{4^2}\cdot\frac{\partial^2 A_2}{\partial x^2} = +\frac{1}{(2.4)^2}\cdot\frac{\partial^4 A_0}{\partial x^4},$$

$$A_6 = -\frac{1}{6^2}\cdot\frac{\partial^2 A_4}{\partial x^2} = -\frac{1}{(2.4.6)^2}\cdot\frac{\partial^6 A_0}{\partial x^6}.$$

. . . . . .

$$A_{2n} = -\frac{1}{(2n)^2}\cdot\frac{\partial^2 A_{2n-2}}{\partial x^2} = \pm\frac{1}{(2.4\ldots 2n)^2}\cdot\frac{\partial^{2n} A_0}{\partial x^{2n}}.$$

If we know the first coefficient A_0, all the others can be deduced. This coefficient A_0 is given by the expression of the potential on the axis, which depends on the form of the layer and on the law of distribution.

The potential outside the axis is thus

$$(11)\quad P = A_0 - \frac{\rho^2}{2^2}\frac{\partial^2 A_0}{\partial x^2} + \frac{\rho^4}{(2.4)^2}\cdot\frac{\partial^4 A_0}{\partial x^4} - \frac{\rho^6}{(2.4.6)^2}\cdot\frac{\partial^6 A_0}{\partial x^6} + \ldots\ldots$$

For the second series, Laplace's equation would have given the general condition

$$\frac{\partial^2 B_{2n+2}}{\partial x^2} + (2n)^2 B_{2n} = 0,$$

which does not enable us to determine the successive coefficients in the same way.

366. In the case of a homogeneous circular layer of radius a and of density equal to unity, the value of the potential P_0 on the axis, taking the centre as origin of the abscissæ, is

$$P_0 = 2\pi\left(\sqrt{a^2+x^2} - x\right).$$

Putting

$$u = \sqrt{a^2+x^2},$$

which gives

$$\frac{\partial u}{\partial x} = \frac{x}{u},$$

we have thus, for the first development of the potential as a function of powers of ρ,

$$A_0 = 2\pi(u - x),$$
$$\frac{\partial A_0}{\partial x} = 2\pi\left(\frac{\partial u}{\partial x} - 1\right) = 2\pi\left(\frac{x}{u} - 1\right),$$
$$\frac{\partial^2 A_0}{\partial x^2} = 2\pi\frac{\partial^2 u}{\partial x^2}, \text{ etc.}$$

The potential at P outside the axis is

$$(12)\quad P = 2\pi\left[u - x - \frac{\rho^2}{2^2}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\rho^4}{(2.4)^2}\cdot\frac{\partial^4 u}{\partial x^4} - \frac{\rho^6}{(2.4.6)^2}\cdot\frac{\partial^6 u}{\partial x^6} + \ldots\right].$$

The successive differentials would be easily calculated.

367. When the layer is circular, it is often more advantageous to carry out the development in another manner.

Let a be the radius of the circle which bounds the layer, r the distance of the point M from the centre of the circle, and θ the angle which the direction of this right line makes with the axis. We may express the potential by one of two series

$$(13)\quad \begin{aligned} P &= A_0 + A_1\left(\frac{r}{a}\right) + A_2\left(\frac{r}{a}\right)^2 + \ldots\ldots, \\ P &= B_0 + B_1\left(\frac{a}{r}\right) + B_2\left(\frac{a}{r}\right)^2 + \ldots\ldots, \end{aligned}$$

according as r is smaller or larger than a—that is to say, that the point M is inside or outside the sphere of radius a. The coefficients are functions of the angle θ, and as the two expressions should have the same value on the sphere, they satisfy the condition

$$A_0 + A_1 + \ldots\ldots = B_0 + B_1 + \ldots\ldots$$

The potential being considered as a function of r and of θ, Laplace's equation becomes

$$r^2\frac{\partial^2 P}{\partial r^2} + 2r\frac{\partial P}{\partial r} + \operatorname{cotan}\theta\frac{\partial P}{\partial \theta} + \frac{\partial^2 P}{\partial \theta^2} = 0.$$

We find thus that the coefficients A and B satisfy the general conditions

$$n(n+1)A_n + \operatorname{cotan}\theta \frac{\partial A_n}{\partial \theta} + \frac{\partial^2 A_n}{\partial \theta^2} = 0.$$

$$n(n-1)B_n + \operatorname{cotan}\theta \frac{\partial B_n}{\partial \theta} + \frac{\partial^2 B_n}{\partial \theta^2} = 0.$$

If we develop the potential on the axis

$$P_0 = 2\pi\left[\sqrt{a^2+x^2} - x\right],$$

as a function of increasing powers of $\frac{x}{a}$ or $\frac{a}{x}$, we obtain the two series

$$P_0 = 2\pi a\left[1 - \frac{x}{a} + \frac{1}{2}\left(\frac{x}{a}\right)^2 - \frac{1.1}{2.4}\left(\frac{x}{a}\right)^4 + \frac{1.1.3}{2.4.6}\left(\frac{x}{a}\right)^6 - \ldots\right],$$

(14)

$$P_0 = 2\pi a\left[\frac{1}{2}\frac{a}{x} - \frac{1.1}{2.4}\left(\frac{a}{x}\right)^3 + \frac{1.1.3}{2.4.6}\left(\frac{a}{x}\right)^5 - \frac{1.1.3.5}{2.4.6.8}\left(\frac{a}{x}\right)^7 + \ldots\right].$$

In order to have the expression for the potential outside the axis as a function of the ratio $\frac{r}{a}$ or of $\frac{a}{r}$, we need only remark that if the density of a spherical layer is symmetrical in reference to a diameter taken about the axis of x, the potential of this layer at a point M only depends on the distance r of this point from the centre O of the sphere, and of the angle θ which the right line OM makes with the axis.

From a well-known theorem of Legendre, this potential may be expressed by the general formulas

$$P = A_0 + A_1X_1\frac{r}{a} + A_2X_2\left(\frac{r}{a}\right)^2 + \ldots\ldots,$$

$$P = B_0\frac{a}{r} + B_1X_1\left(\frac{a}{r}\right)^2 + B_2X_2\left(\frac{a}{r}\right)^3 + \ldots\ldots,$$

in which A_0, A_1 ... B_0, B_1 ..., are constants, and X_1, X_2 ... functions of the angles θ known as Legendre's polynomials, and which are defined by the series

$$[1 - 2x\cos\theta + x^2]^{-\frac{1}{2}} = 1 + X_1 x + X_2 x^2 + X_3 x^3 + \ldots\ldots;$$

all these functions become equal to unity when the angle θ is equal to zero.

As we know the development of the potential of a homogeneous circular layer for a point of the axis—that is to say, when θ is zero and r equal to x, the coefficients are known. It follows that the potential P for a point outside the axis is expressed by

$$P = 2\pi a\left[1 - X_1\frac{r}{a} + \frac{1}{2}X_2\left(\frac{r}{a}\right)^3 - \frac{1.1}{2.2}X_4\left(\frac{r}{a}\right)^4 + \frac{1.1.3}{2.4.6}X_6\left(\frac{r}{a}\right)^6 - \ldots\right],$$

(15)

$$P = 2\pi a\left[\frac{1}{2}\frac{a}{r} - \frac{1.1}{2.4}X_2\left(\frac{a}{r}\right)^3 + \frac{1.1.3}{2.4.6}X_4\left(\frac{a}{r}\right)^5 - \frac{1.1.3.5}{2.4.6.8}X_6\left(\frac{a}{r}\right)^7 + \ldots\right].$$

368. POTENTIAL OF A UNIFORM CIRCULAR SHELL. — The potential of a uniform circular shell may now be obtained by the expression

$$V = -\Phi\frac{\partial P}{\partial x}.$$

We thus find, with the first form (12),

$$V = 2\pi\Phi\left[1 - \frac{\partial u}{\partial x} + \frac{\rho^2}{2^2}\cdot\frac{\partial^3 u}{\partial x^3} - \frac{\rho^4}{(2.4)^2}\cdot\frac{\partial^5 u}{\partial x^5} + \ldots\ldots\right].$$

The first terms of the development are then

(16)

$$V = 2\pi\Phi\left\{1 - \frac{x}{u}\left[1 + \frac{3}{2^2}\frac{a^2}{u^2}\cdot\left(\frac{\rho}{u}\right)^2 - \frac{3.5}{(2.4)^2}\cdot\frac{4x^2 - 3a^2}{u^2}\cdot\left(\frac{\rho}{u}\right)^4 + \ldots\right]\right\}$$

and the series is convergent whenever $\rho < u$.

Taking the expression

$$V = \Phi \frac{\partial P}{\partial a},$$

the second form of the development will give

$$V = 2\pi\Phi\left[1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 X_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{a}\right)^4 X_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{r}{a}\right)^6 X_6 + \ldots\right],$$

(17)

$$V = 2\pi\Phi\frac{a}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 X_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{r}\right)^4 X_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{a}{r}\right)^6 X_6 + \ldots\right].$$

369. POTENTIAL OF A SPHERICAL LAYER.—Let us finally consider any given spherical layer of radius a. The potential at an external point M, at a distance r from the centre, may still be expressed by the series

$$(18) \qquad V = a\left[A_0\left(\frac{a}{r}\right) + A_1\left(\frac{a}{r}\right)^2 + A_2\left(\frac{a}{r}\right)^3 + \ldots\ldots\right],$$

in which the coefficients depend on the law of distribution and on the direction of the right line r.

Let u be the angle of the right line r with the axis of z, l the angle of the plane rz with the plane of yz; the co-ordinates of the point M are then

$$(19) \qquad \begin{aligned} z &= r\cos u, \\ y &= r\sin u\cos l, \\ x &= r\sin u\sin l. \end{aligned}$$

Taking r, u and l as independent variables, Laplace's equation gives

$$r\frac{\partial^2(rV)}{\partial r^2} + \frac{\partial^2 V}{\partial u^2} + \operatorname{cotan} u\frac{\partial V}{\partial u} + \frac{1}{\sin^2 u}\cdot\frac{\partial^2 V}{\partial l^2} = 0;$$

from which follows the general condition for the coefficients

$$n(n+1)A_n+\frac{\partial^2 A_n}{\partial u^2}+\operatorname{cotan} u\,\frac{\partial A_n}{\partial u}+\frac{1}{\sin^2 u}\cdot\frac{\partial^2 A_n}{\partial l^2}=0.$$

The general integral of this equation was given by Laplace; if we put

$$A_{n\cdot m}=\left[\cos^{n-m}u-\frac{(n-m)(n-m-1)}{2(2n-1)}\cos^{n-m-2}u\right.$$

$$\left.+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4\cdot(2n-1)(2n-2)}\cos^{n-m-4}u-\ldots\right]\sin^m u,$$

the coefficient A_n, expressed by means of the new symbols $A_{n\cdot m}$, consists of $2n+1$ terms developed according to the sines and cosines of multiples of the angles l, and its value is

$$A_n=g_{n\cdot 0}A_{n\cdot 0}+(g_{n\cdot 1}\cos l+h_{n\cdot 1}\sin l)A_{n\cdot 1}$$

$$+(g_{n\cdot 2}\cos 2l+h_{n\cdot 2}\sin 2l)A_{n\cdot 2}+\ldots+(g_{n\cdot n}\cos nl+h_{n\cdot n}\sin nl)A_{n\cdot n}.$$

The factors denoted by g, h, with the different indices, are numerical coefficients which must be determined in each special case.

370. If we consider a sphere magnetised in any given way, its external action is equal to that of two layers of equal mass and contrary signs, distributed on the surface according to a certain law.

The coefficient A_0 of the first term is null. For, in fact, at a great distance, the potential simply becomes equal to the quotient of the total mass by the distance. The product A_0a^2 which forms the numerator of the first term represents in this case the total mass, and we know that in every magnet the total mass is null.

The value of the coefficient A_1 of the next term is

$$A_1=g_{1\cdot 0}\cos u+(g_{1\cdot 1}\cos l+h_{1\cdot 1}\sin l)\sin u,$$

or, taking equations (19) into account,

$$A_1=g_{1\cdot 0}\frac{z}{r}+g_{1\cdot 1}\frac{y}{r}+h_{1\cdot 1}\frac{x}{r}.$$

This term becomes predominant at a great distance, and the potential reduces then to

$$V = A_1 \frac{a^3}{r^2} = \frac{a^3 g_{0.1}}{r^2}\frac{z}{r} + \frac{a^3 g_{1.1}}{r^2}\frac{y}{r} + \frac{a^3 h_{1.1}}{r^2}\frac{x}{r};$$

it follows from this (151) that the three products $a^3 g_{0.1}$, $a^3 g_{1.1}$, $a^3 h_{1.1}$ represent respectively the magnetic moments of the sphere in reference to the axes of x, of y, and of z. Denoting by $a^3 K$ the resultant magnetic moment, and by α, β and γ the cosines of the angles which its direction makes with the axes, we have

$$K = \frac{h_{1.1}}{\alpha} = \frac{g_{1.1}}{\beta} = \frac{g_{1.0}}{\gamma} = \sqrt{h_{1.1}^2 + g_{1.1}^2 + g_{1.0}^2}.$$

371. SOLENOIDAL MAGNETS.—The potential of a solenoidal magnet (330) only depends on the surfaces formed by the ends of the elementary solenoids which constitute it.

If all the solenoids are closed, the potential of the magnet is everywhere null, and the magnetic force null. In this case the induction is reduced at each point to $4\pi I$, and it is parallel to the magnetisation.

372. Suppose that a solenoidal magnet is bounded by a channel surface, the magnetisation being everywhere normal to the perpendicular section of the channel. The flow of induction across an element dS of the right section is equal to $4\pi I dS$, and the value of the flow of induction is

$$Q = 4\pi \int I dS.$$

Each of the solenoidal filaments forms a closed curve of length l, perpendicular at every point to the right section of the channel. If the structure of the magnet is such that the product of the strength of magnetisation I of a filament, by its length l, is a constant quantity, examples of which we shall see afterwards, the flow of induction could be expressed by the formula

$$(20) \qquad Q = 4\pi \int I dS = 4\pi A \int \frac{dS}{l}.$$

If the magnet is a ring of revolution, and x is the radius of an elementary solenoid, we shall have

$$Q = 4\pi A \int \frac{dS}{2\pi x} = 2A \int \frac{dS}{x}.$$

Consider a torus or anchor ring, for instance. Let a be the radius of the section and R the distance of its centre from the axis of rotation taken as axis of z; we have then

$$\int \frac{dS}{x} = \int_{R-a}^{R+a} \frac{z}{x} dx = \pi \left[R - \sqrt{R^2 - a^2} \right]$$

and the value of the total flow of induction is

$$Q = 2\pi A \left[R - \sqrt{R^2 - a^2} \right]. \tag{21}$$

373. CYLINDER.—A cylinder uniformly magnetised and terminated by right sections, is equivalent to two equal and opposite magnetic layers $\pm I$, which cover the two bases A and B. The potential of any such magnet at a given point is equal to the sum of the potentials V_a and V_b, of the two terminal layers.

If the right section of the cylinder is circular, the potentials V_a and V_b may be expressed by the formulæ found previously (**365** and **366**).

The expression for the magnetic force on a point M of the axis on the outside and on the side of the positive face A, is

$$F = 2\pi I(1 - \cos\alpha) - 2\pi I(1 - \cos\beta) = 2\pi I(\cos\beta - \cos\alpha),$$

α and β being the angles under which the radii of the two bases are seen from the point M, and it is in the same direction as that of the magnetisation.

For a point in the interior, the actions of the two bases are of the same sign, which gives a force

$$F = 4\pi I - 2\pi I(\cos\alpha + \cos\beta),$$

in the opposite direction of that of the magnetisation.

Lastly, the induction on the axis in the interior is

$$F_1 = 4\pi I - F = 2\pi I(\cos\alpha + \cos\beta);$$

it is parallel to the magnetisation and varies very slowly so long as the point in question is at a considerable distance from the bases.

If l is the length of the cylinder and a its radius, the value of the induction F_0 at the centre of the cylinder is

$$F_0 = 2\pi I \,.\, 2\cos\alpha = 4\pi I \frac{\frac{l}{2}}{\sqrt{a^2 + \left(\frac{l}{2}\right)^2}} = 4\pi I \frac{1}{\sqrt{1 + \left(\frac{2a}{l}\right)^2}}$$

When the length of the cylinder is very great as compared with its diameter, we may take the approximate expression

$$F_0 = 4\pi I\left[1 - \frac{1}{2}\left(\frac{2a}{l}\right)^2\right].$$

The induction is then sensibly the same throughout the whole extent of the median section, and the expression for the total flow of induction Q_0 which traverses it is

$$Q_0 = \pi a^2 F_0 = (2\pi a)^2 I\left[1 - \frac{1}{2}\left(\frac{2a}{l}\right)^2\right].$$

If the cylinder is so long that the quantity in brackets does not sensibly differ from unity, the flow of induction in the median section is equal to $(2\pi a)^2 I$. This flow is proportional therefore to the square of the contour; it has sensibly the same value in any given section sufficiently distant from the ends.

CHAPTER IV.

MAGNETIC INDUCTION.

374. General Characteristics of Magnetic Induction.—There is probably no substance which, when placed in a magnetic field, does not experience the effect of induction—that is to say, does not itself become a magnet, at any rate temporarily.

When the body is isotropic, the axis of induced magnetisation coincides everywhere with the direction of the magnetic force. In certain bodies the induced magnetisation is in the same direction as the force; these are the bodies which we have called *paramagnetic* or simply *magnetic*. In others the direction of the magnetisation is opposite that of the force; these bodies are *diamagnetic*. In the presence of a pole of a magnet, the nearest part of bodies of the first class acquires polarity of the opposite kind; bodies of the second class acquire a pole of the same kind.

We shall assume that at every point of an isotropic body submitted to magnetic induction, the magnetisation is proportional to the resultant of all the magnetic forces which are exerted at this point. These forces depend not only on the original field, but also on the magnetism developed by induction on the body itself. If F is the resultant force, to which the name *magnetising force* is sometimes given, I the intensity of magnetisation, we may write

$$(1) \qquad I = kF.$$

The factor k, which expresses the ratio of the magnetisation to the magnetising force, is called the *coefficient of induced magnetisation;* this coefficient is positive or negative, according as the body is magnetic (in the ordinary sense of the word) or is diamagnetic.

The hypothesis of the proportionality of the magnetisation to the magnetising force is verified with close approximation whenever k

has a very small value. This is the case with most magnetic substances, with the exception of iron, nickel, and cobalt. In the case of iron or nickel, for which the coefficient k reaches very high values, such as 30 or 40, proportionality exists as long as the force F does not exceed a certain limit; when the bodies are magnetised by the earth, for example. This is also the case with ordinary iron, twisted iron, cast iron, and steel more or less tempered, the coefficient of magnetisation of which is considerably weaker. The coefficient k is always very small for diamagnetic bodies; it scarcely amounts to $\frac{1}{400,000}$ for bismuth, which is the most active body of this second class.

If the proportionality between the magnetisation and the magnetic force does not exist, we may consider the coefficient k as being itself a function of magnetisation. We shall first investigate the case in which this coefficient is constant and the same in all directions—that is to say, in which the body is isotropic and the induced magnetisation somewhat feeble.

375. INDUCED MAGNETISATION IS PROPORTIONAL TO THE MAGNETISING FORCE.—Consider any given body in the magnetic field. Let V be the potential of the field and Ω that which is produced by induced masses, the value of the actual potential U will be

$$U = V + \Omega.$$

At any given point the components of the magnetising force parallel to the axis are

$$X = -\frac{\partial U}{\partial x}, \quad Y = -\frac{\partial U}{\partial y}, \quad Z = -\frac{\partial U}{\partial z}.$$

The expression for the force itself is

$$F = -\frac{dU}{dn};$$

its direction is that of the perpendicular n, to the equipotential surface which passes through the point in question.

The intensity of magnetisation I, and its components A, B, and C parallel to the axes then become

$$I = kF = -k\frac{dU}{dn},$$

$$A = -k\frac{\partial U}{\partial x}, \quad B = -k\frac{\partial U}{\partial y}, \quad C = -k\frac{\partial U}{\partial z}.$$

Adding these last equations after having multiplied them by dx, dy, dz respectively, we get

$$Adx + Bdy + Cdz = -k\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = -kdU.$$

If the value of k is constant throughout the body, the first member of the equation is the exact differential of a function Φ of the coordinates, and we have

$$A = \frac{\partial \Phi}{\partial x}, \quad B = \frac{\partial \Phi}{\partial y}, \quad C = \frac{\partial \Phi}{\partial z}.$$

It follows from this (332) that the magnetisation is *lamellar*.

376. The Induced Magnetisation is Superficial.—On the other hand the general expression for magnetic density

$$\rho = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)$$

here reduces to

$$\rho = -k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) = -k\Delta U.$$

Since, from Poisson's equation, we have $\Delta U = 4\pi\rho$, we get

$$\rho(1 + 4\pi k) = 0, \qquad \text{or} \qquad \rho = 0;$$

that is to say that *the magnetic density is zero throughout the whole extent of the body*. The magnetisation is then also *solenoidal*, and there is no free magnetism except on the bounding surface of the body.

This conclusion presupposes that the parenthesis $1+4\pi k$ is not zero; but this latter case never occurs, the absolute value of k for diamagnetic bodies being far from attaining $\frac{1}{4\pi}$.

377. The surface density σ of the induced layer is

$$\sigma = \mathrm{I}\cos\theta,$$

θ being the angle of the magnetisation with the perpendicular to the surface (Fig 84).

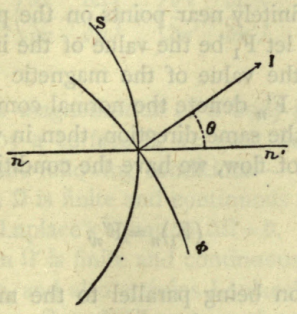

Fig. 84.

Let this perpendicular be called n when it is drawn inwards, and n' when it is drawn outwards, and let a be the perpendicular to the surface for which the function Φ has a constant value, we have

$$\mathrm{I} = \frac{d\Phi}{da},$$

$$\sigma = \mathrm{I}\cos\theta = \frac{\partial\Phi}{\partial n'} = -\frac{\partial\Phi}{\partial n}.$$

378. The value of the potential due to the induced masses—that is to say, to the surface layer—is

$$\Omega = \int\int \frac{\sigma d\mathrm{S}}{r}.$$

The function Ω is finite and continuous, and satisfies Laplace's

equation, both inside and outside the surface. If we denote by Ω' its value on the outside, for two infinitely near points on opposite sides of the surface, we shall have the condition

$$\Omega = \Omega'.$$

379. EQUATION OF CONTINUITY.—COEFFICIENT OF INDUCTION. —The principle of the conservation of the flow of induction (323) enables us to establish in a very simple manner the conditions of continuity V, U, and Ω, at the surface of the magnetised body.

Consider two infinitely near points on the perpendicular on each side of the surface; let F_1 be the value of the induction at the point n the interior, F'_1 the value of the magnetic force at the external point. If $(F_1)_n$ and F'_n denote the normal components of these two forces calculated in the same direction, then in virtue of the theorem of the conservation of flow, we have the condition

$$(F_1)_n = F'_n.$$

The magnetisation being parallel to the magnetising force, the induction in the present case becomes

$$F_1 = F + 4\pi I = F + 4\pi k F = F(1 + 4\pi k);$$

it is proportional to the magnetising force.

If we put

$$\mu = 1 + 4\pi k,$$

we have

$$F_1 = \mu F,$$

and the equation relative to the surface becomes then

$$(2) \qquad \mu F_n = F'_n, \qquad \text{or} \qquad \mu = \frac{F'_n}{F_n}.$$

Thus, for *two infinitely near points on either side of the surface, the ratio of the perpendicular components of the magnetic force is constant.* This is a fundamental deduction from Poisson's theory, which we have already used (111) in defining dielectrics. The coefficient μ

represented the specific inductive capacity of electricity; we shall here call it the *coefficient of magnetic induction.* It must not be confounded with the coefficient of induced magnetism which has been represented by k.

380. Expressing equation (2) as a function of the potentials, we get

$$\mu \frac{\partial U}{\partial n} + \frac{\partial U'}{\partial n'} = 0,$$

or

$$\mu \left(\frac{\partial V}{\partial n} + \frac{\partial \Omega}{\partial n}\right) + \left(\frac{\partial V}{\partial n'} + \frac{\partial \Omega'}{\partial n'}\right) = 0. \tag{3}$$

To determine the magnetisation of a body placed in a magnetic field, and bounded by a surface S, we must find two conditions Ω and Ω' which satisfy the following conditions—

1st. The function Ω is finite and continuous in the interior of the surface, and satisfies Laplace's equation $\Delta\Omega = 0$.

2nd. The function Ω' is finite and continuous on the exterior, zero at an infinite distance, and also satisfies Laplace's equation.

3rd. The functions Ω and Ω' are equal to each other on the surface, and their differentials satisfy the equation of continuity (3).

These functions represent the potential of a magnetic layer distributed on the surface of the body. The density of this layer at every point is determined by the variation of the normal components, which gives

$$F'_n - F_n = 4\pi\sigma;$$

from which is deduced

$$\sigma = \frac{1}{4\pi}(F'_n - F_n) = \frac{\mu - 1}{4\pi} F_n = kF_n.$$

381. Case of Two Different Magnetic Media.—Relative Magnetisation.—Let us suppose that the body A, bounded by the surface S, is situated in a magnetic medium whose coefficient of induction is μ'; the theorem of the conservation of the flow of induction gives still

$$\mu F_n = \mu' F'_n,$$

that is to say

$$\mu \frac{\partial U}{\partial n} + \mu' \frac{\partial U'}{\partial n'} = 0,$$

or

$$\mu \left(\frac{\partial V}{\partial n} + \frac{\partial \Omega}{\partial n} \right) + \mu' \left(\frac{\partial V}{\partial n'} + \frac{\partial \Omega'}{\partial n'} \right) = 0. \tag{4}$$

The functions Ω and Ω' which determine the surface layer are defined by the same conditions as the preceding, with this single difference, that the equation of continuity (4) contains the coefficients of induction of the two media.

The surface density is still given by the perpendicular components

$$4\pi\sigma = F'_n - F_n = F_n \left(\frac{\mu}{\mu'} - 1 \right).$$

If we put

$$\mu_1 = \frac{\mu}{\mu'} = 1 + 4\pi k_1,$$

that is to say

$$k_1 = \frac{\mu - \mu'}{4\pi\mu'} = \frac{k - k'}{1 + 4\pi k'},$$

the expression for the density becomes

$$4\pi\sigma = F_n(\mu_1 - 1), \qquad \text{or} \qquad \sigma = k_1 F_n.$$

It is this surface layer which determines the motion of the body A in the medium. It is the same as if the external medium were suppressed, or more exactly replaced by air, and the coefficient of induction of the body replaced by another value μ_1, or the coefficient of magnetisation k by a different value k_1. The apparent magnetisation I_1, of the body would thus have for its perpendicular projection

$$I_1 \cos\theta = \sigma = \frac{1}{4\pi} (\mu_1 - 1) F_n = k_1 F_n.$$

382. The discussion of this problem gives rise to some conclusions analogous to those which are deduced from the principle of Archimedes for bodies immersed in liquids.

We may, in fact, consider k_1 as the relative coefficient of magnetisation of a body, in reference to the medium which surrounds it, k and k', being coefficients of the two media in respect of air. If the coefficient k of the body is greater than the coefficient k' of the medium, the value of k_1 is positive, and the apparent magnetisation of the body is positive. If, on the contrary, $k<k'$, the value of k_1 is negative and the body will appear diamagnetic. When the coefficients k and k' are equal, the magnetisation of the body A will appear to be null, which ought to be the case, for it is in a medium identical with itself, and the induced magnetism is superficial.

We are thus led to assume that there is no real opposition of properties between magnetic and diamagnetic bodies, and that the difference of the effects is due to the greater or less magnetic character of the external medium. As diamagnetic bodies retain their characteristic properties in the most perfect vacuum which has been produced, we must assume, on this view, that a vacuum is a magnetic medium, and that its coefficient of magnetisation is greater in absolute value than that of all known diamagnetic substances.

If, on the contrary, we assume the value zero for the coefficient of magnetisation of vacuum, a negative value must be assigned to those of all diamagnetic bodies. In this case the coefficient of induction μ is greater than unity for magnetic bodies, and is less than unity for diamagnetic bodies. No body is known in which μ is negative since the coefficient of diamagnetic bodies is never greater than $\frac{1}{4\pi}$ in absolute value; we have already said that for bismuth, the most diamagnetic of all known substances, k is about $= -\frac{1}{400{,}000}$. The coefficient of induction of diamagnetic bodies only differs from unity by an infinitely small quantity. For soft iron and nickel the coefficient k being comprised between 30 and 40, the value of μ is nearly 500. The ratio of the two absolute values of k for iron and bismuth is then almost

$$40 \times 400{,}000 = 1{,}6 \,.\, 10^7.$$

We may however remark that the influence of a magnetic medium could only be exactly compared with that of a fluid, and the principle of Archimedes be applied, provided that $k_1 = k - k'$. From the preceding remark it appears that this ratio is very nearly verified for all diamagnetic bodies, and for those also which are very slightly magnetic; but it would be far from the truth if the surrounding medium had a coefficient of magnetisation near unity,

and especially if the magnetic properties of the medium were comparable with those of soft iron.

383. MAGNETIC SUSCEPTIBILITY AND PERMEABILITY. — The phenomena of magnetic induction may, as we have seen, be expressed with the aid of two coefficients.

The *coefficient of induced magnetisation k* expresses the ratio of the intensity of magnetisation to the magnetic force; in other words, the intensity of magnetisation in a field equal to unity. This coefficient is sometimes known as that of Neumann, who first used it. Sir W. Thomson calls it the *coefficient of magnetic susceptibility.*

The second coefficient called μ, is the *coefficient of magnetic induction;* it is the analogue of the specific inductive capacity of a dielectric in electricity. It is equal to the ratio of the perpendicular components of the force on the outside and inside of the medium in question, and is connected with the preceding by the ratio

$$\mu = 1 + 4\pi k.$$

Sir W. Thomson calls this coefficient μ, *the coefficient of magnetic permeability.* The following are his reasons for using this expression—

"The analogue corresponding to conducting power of a solid for heat—or, as it is shortly called, 'thermal conductivity'—is, in electrostatic induction, the 'specific inductive capacity' of the dielectric; in magnetism it is not what has hitherto been called magnetic inductive capacity—a quality which is negative in diamagnetics—but it is Faraday's 'conducting power for lines of force,' and in hydrokinetics it is flux per unit area, per unit intensity of energy. The common word *permeability* seems well adapted to express the specific quality in each of the four analogous subjects. Adopting it we have thermal permeability, a synonym for thermal conductivity; permeability for lines of electric force a synonym for the electrostatic inductive capacity of an insulator; magnetic permeability a synonym for conducting power for lines of magnetic force; and hydrokinetic permeability a name for the specific quality of a porous solid according to which, when placed in a moving frictionless liquid, it modifies the flow."—(*Reprint of Papers*, §628.)

384. ANISOTROPIC BODIES.—We have hitherto only considered the induction produced in isotropic bodies. The experiments of Plücker on bodies with a fibrous texture, and on crystals, have shown that magnetic action is exerted unequally in different directions. Poisson had predicted the existence of such bodies, and in order to explain them on his theory we must substitute for the conducting

spheres, disseminated in a non-conducting medium, equal ellipsoids turned in the same direction. If the body thus constituted "were a homogeneous sphere, and it were made to turn without displacing its centre of gravity, and without any change either in the external forces or in the function V, the magnitude and directions of the magnetic forces in this body would nevertheless change. As this particular case has not yet been met with, we shall for the present exclude it from our researches."—(*Mémoire sur la théorie du magnétisme*, "Mém. de l'Institut pour 1821–22," Vol. v., p. 278.)

For the reasons which we have already developed in electricity (215), there must in this case be three principal axes of magnetisation. If we place each of these axes successively in the direction of the field, we shall have between the coefficients k, k', k'' of susceptibility, and the coefficients μ, μ', μ'' of permeability, the ratios

$$\mu = 1 + 4\pi k, \qquad \mu' = 1 + 4\pi k', \qquad \mu'' = 1 + 4\pi k''.$$

When the body is directed in any manner whatever in reference to the field, we may substitute for the true field three fields whose directions are rectangular and parallel to the principal axes, and consider the real magnetisation as the resultant of these three magnetisations.

This superposition is evidently legitimate in all cases of diamagnetic, or of slightly magnetic bodies; it is only a consequence of the principle of the proportionality of the magnetisation to the magnetising force.

385. Uniform Magnetisation.—If the surface of a body is such that uniform magnetisation in a certain direction produces a certain constant force, and we place it in a uniform field, the force of which is parallel to this direction, it will acquire a uniform magnetisation, since the magnetising force will have the same value at all points, and that, without its being necessary to make any restriction on the magnitude of the coefficient k. The magnetising force F being the sum of the strength of the field ϕ, and of a force CI due to induced magnetisation, which is evidently proportional to the intensity of the magnetisation, we shall have

$$\mathrm{I} = k\,\mathrm{F} = k(\phi + \mathrm{CI}),$$

and therefore

$$\mathrm{I} = \frac{k}{1 - k\mathrm{C}}\phi.$$

Since the internal action of a uniformly magnetised body is given (354) by the partial differentials of the second order of the potential P of a homogeneous mass, the force can only be constant provided the function P is represented by a polynomial of the second degree—that is to say, if the body is bounded by a surface of the second degree. When the coefficient k is very small—that is to say, for all diamagnetic and feebly magnetic bodies—we have sensibly

$$I = k\phi.$$

In this case the magnetisation induced in a uniform field is under the same laws of induction as are dielectrics, so that all the results to which we have attained in electrostatics are also applicable to magnetism, without any other modification than the substitution of the magnetic potential for the electrical potential, and of the coefficient of magnetic induction for the specific inductive capacity.

386. SPHERE.—For a uniformly magnetised sphere (355) we have

$$C = -\frac{4}{3}\pi.$$

The magnetisation produced on a sphere by a uniform field will be

$$I = \frac{k}{1 + \frac{4}{3}\pi k}\phi = \frac{3}{4\pi}\frac{\mu - 1}{\mu + 2}\phi = \frac{3}{4\pi}h\phi.$$

The coefficient h or $\frac{\mu - 1}{\mu + 2}$ is equal to unity for conductors of electricity; it is always positive and less than unity for magnetic bodies, and it differs little from unity when μ is very great. This coefficient, on the contrary, is negative and very small for diamagnetic bodies.

The value of the magnetic moment of the sphere is

$$\varpi = uI = hR^3\phi.$$

The resultant force within the sphere is

$$F = \phi - \frac{4}{3}\pi I = \phi(1 - h) = \phi\frac{3}{\mu + 2},$$

and the induction

$$F_1 = F\mu = \phi\frac{3\mu}{\mu+2}.$$

The former of these expressions also represents the total external force on a point near the equator, and the second on a point near the pole. The ratio of these two forces is therefore equal to $\frac{1}{\mu}$.

In the case in which μ is very great, the formulæ become simpler, and we have sensibly

$$F = \frac{3}{\mu}\phi,$$
$$F_1 = 3\phi,$$
$$\varpi = R^3\phi.$$

387. Poisson's Hypothesis.—If we suppose with Poisson that a magnetic body is made up of a system of small spheres, which are absolute magnetic conductors ($\mu = \infty$), disseminated in a non-magnetic medium, the ratio of the volume occupied by all the spheres, to the total volume, is expressed (167) by

$$k = \frac{\mu - 1}{\mu + 2}.$$

Taking the value $\mu = 500$ for iron, we get

$$k = 1 - \frac{3}{500} = 1 - \frac{1}{167}.$$

But the maximum value which the ratio k can have with equal spheres is $\frac{\pi}{3\sqrt{2}} = 1 - \frac{1}{3.86}$. We must therefore suppose that in the present case the volumes of the spheres are not the same, and that the greater intervals are occupied by spheres of smaller diameter. It appears, however, difficult to suppose that the adjacent spheres do not act on each other, and that the magnetisation of each of them, as is assumed on Poisson's theory, could be solely dependent on the external field.

388. Ellipsoid. — Cylinder. — For an ellipsoid magnetised uniformly in any given direction, the components of the internal force parallel to the axes are equal respectively to $-AL$, $-BM$, and $-CN$ (356).

In a uniform field in which the force ϕ makes with the axes angles whose cosines are λ, λ', λ'', the components of the magnetisation will be

$$A = \frac{k}{1 + kL}\phi\lambda,$$

$$B = \frac{k}{1 + kM}\phi\lambda',$$

$$C = \frac{k}{1 + kN}\phi\lambda''.$$

These equations presuppose that the magnetisation is so weak that we may admit the effects produced in different directions to be superposed.

If one of the axes of the ellipsoid is parallel to the direction of the field—the axis of x for instance—we have

$$I = \frac{k}{1 + kL}\phi,$$

and however great may be the value of k, the magnetisation will be uniform.

By the results indicated in (357) we might apply this expression to several different cases.

For an unlimited cylinder, perpendicular to the direction of the field (358), we shall have

$$I = \frac{k}{1 + 2\pi k}\phi.$$

389. Barlow's Problem.—The case of a dielectric layer comprised between the surfaces of two concentric spheres, and placed in a uniform field (166) corresponds to that of a magnetic layer of the same form placed in the same conditions. This question is known as *Barlow's problem.*

There are produced then, as we have seen in the case of electricity (165), two magnetic layers, one M_1 on the internal surface S_1 (Fig. 85), and the other M_2 on the external surface S_2 of the volume in question.

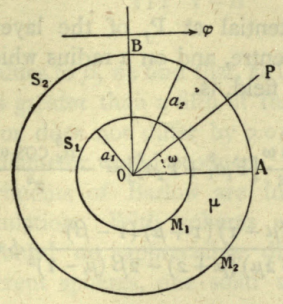

Fig. 85.

The internal actions F_1 and F_2 of these two layers are determined by the equations

$$\frac{F_1}{3(\mu-1)} = \frac{-F_2}{(\mu-1)[1+2\mu-2(\mu-1)\beta]} = \frac{\phi}{9\mu+2(\mu-1)^2(1-\beta)},$$

where β denotes the ratio $\left(\frac{a_1}{a_2}\right)^3$ of the cubes of the radii.

The action of these two layers is constant in the interior of the small sphere S_1, and its value is

$$F = \frac{\phi}{1+\frac{2(\mu-1)^2}{9\mu}(1-\beta)},$$

but it is not constant within the magnetic substances, nor without.

We have then, for the total action $\phi - F$ on the interior,

$$\frac{\phi-F}{F} = \frac{2(\mu-1)^2}{9\mu}(1-\beta) = \frac{8\pi k}{9\left(\frac{1}{4\pi k}+1\right)}(1-\beta),$$

and, if k is very great,

$$\frac{\phi - F}{F} = \frac{8}{9}\pi k(1-\beta).$$

The external potential at P, of the layers M_1 and M_2, at a distance r from the centre, and on a radius which makes the angle ω with the force of the field, is

$$V_e = -\frac{\cos\omega}{r^2}[F_1 a_1^3 + F_2 a_2^3] = -\frac{a_2^3\cos\omega}{r^2}[F_1\beta + F_2]$$

$$= \frac{a_2^3\cos\omega}{r^2}\frac{(\mu-1)(1+\mu)(1-\beta)}{(1+2\mu)(\mu+2)-2\beta(\mu-1)^2}\phi = A\frac{a_2^3\cos\omega}{r^2}\phi,$$

putting

$$A = \frac{(\mu-1)(1+2\mu)(1-\beta)}{(\mu+2)(1+2\mu)-2\beta(\mu-1)^2};$$

from which is deduced, for the action of the induced magnetism in two points at a distance r on the line OA and on the line OB, that is at the pole and at the equator,

$$X_p = 2\left(\frac{a_2}{r}\right)^3 A\phi,$$

$$X_e = \left(\frac{a_2}{r}\right)^3 A\phi.$$

390. For a solid sphere we have $\beta = 0$, and therefore

$$A_0 = \frac{(\mu-1)(1+2\mu)}{(1+2\mu)(\mu+2)} = \frac{\mu-1}{\mu+2} = k.$$

The ratio of the actions excited at the same point by a hollow sphere, and by a solid one of the same external diameter, is then

$$\frac{A}{A_0} = \frac{(\mu+2)(1+2\mu)(1-\beta)}{(\mu+2)(1+2\mu)-2\beta(\mu-1)^2} = \frac{1}{1+\dfrac{9\mu}{(\mu+2)(1+2\mu)}\cdot\dfrac{\beta}{1-\beta}}.$$

Applying this formula to iron, and taking $\mu = 500$, we get

$$\frac{A}{A_0} = \frac{1}{1 + \frac{1}{112} \cdot \frac{\beta}{1-\beta}}.$$

By giving different values to β, we find that, as long as the thickness of the spherical layer is greater than a fifth of the radius, the magnetic action on the exterior does not differ by 0.01 of that which a solid sphere of the same diameter would produce.

The older experiments of Barlow are in conformity with the results of this calculation. With spheres of 10 inches external diameter, Barlow found no appreciable difference between the actions of two different spheres, one solid and the other hollow, the thickness of the latter being equal to $\frac{2}{3}$ of the radius.

On the other hand, the action of the hollow sphere was only $\frac{2}{3}$ that of the solid one, when the thickness was reduced to about the $\frac{1}{30}$ of an inch. Taking the value of μ at 500, calculation would give about $\frac{1}{2}$ for the ratio of the two actions.

391. We have seen (386) that the total action in the vicinity of a solid sphere is

$$F_p = \phi \frac{3\mu}{\mu + 2},$$

near the pole, and

$$F_e = \phi \frac{3}{\mu + 2}$$

ear the equator. Still taking $\mu = 500$, as for soft iron, we get

$$F_p = 3\phi \frac{1}{1 + \frac{2}{500}} = 3\phi \left(1 - \frac{2}{250}\right),$$

$$F_e = 3\phi \frac{1}{502} = 3\phi \frac{1}{500}, \text{ nearly.}$$

The force is sensibly zero at the equator, and at the pole its value is three times that of the force of the field.

If the coefficient μ is near unity, and we put

$$\mu = 1 + \alpha,$$

we get in like manner

$$F_p = \phi \frac{1+\alpha}{1+\frac{\alpha}{3}} = \phi\left(1+\frac{2\alpha}{3}\right),$$

$$F_e = \phi \frac{1}{1+\frac{\alpha}{3}} = \phi\left(1-\frac{\alpha}{3}\right).$$

In the interior of a hollow sphere, the force is

$$F = \phi \frac{1}{1+\frac{2(\mu-1)^2}{9\mu}.(1-\beta)}.$$

If the coefficient μ is very great, we may write

$$F = \phi \frac{1}{1+\frac{2}{9}(\mu-2)(1-\beta)}.$$

If the coefficient μ is very near unity, and we again put $\mu = 1+\alpha$, we have

$$\frac{F}{\phi} = \frac{1}{1+\frac{2}{9}\alpha^2(1-\beta)} = 1-\frac{2}{9}\alpha^2(1-\beta).$$

A small magnetised needle introduced into the sphere would determine there a new induced layer, which would be superposed on the first, and the distribution of which can be easily calculated, for the external action of such a needle is equal to that of a uniformly magnetised sphere; but the action of this new layer will always be parallel to the magnetic axis of the needle, and will not influence its direction. The oscillations of this needle depend then only on the resultant actions of the external field, and on the layers induced by the field itself.

392. Anisotropic Bodies.—Consider an anisotropic body in a uniform field. Let k, k', k'' be the three principal coefficients of magnetisation, and λ, λ', λ'' the cosines of the angles of the strength of the field with the axes; the coefficients of magnetisation being supposed to be very small, the values of the intensities of the three partial magnetisations will be

$$\begin{aligned} \mathrm{I} &= k\lambda\phi, \\ \mathrm{I}' &= k'\lambda'\phi, \\ \mathrm{I}'' &= k''\lambda''\phi; \end{aligned}$$

and the corresponding magnetic moments

$$\begin{aligned} m &= u\mathrm{I} = u\phi k\lambda, \\ m' &= u\mathrm{I}' = u\phi k'\lambda', \\ m'' &= u\mathrm{I}'' = u\phi k''\lambda''. \end{aligned}$$

From this we deduce, for the resultant magnetic moment,

$$\mathrm{M}^2 = m^2 + m'^2 + m''^2 = u^2\phi^2(k^2\lambda^2 + k'^2\lambda'^2 + k''^2\lambda''^2) = u^2\phi^2\mathrm{H}^2.$$

The resultant magnetic axis of the sphere makes with the axes of the co-ordinates, angles whose cosines α, α', α'' are defined by the equations

$$\frac{\alpha}{k\lambda} = \frac{\alpha'}{k'\lambda'} = \frac{\alpha''}{k''\lambda''} = \frac{1}{\mathrm{H}},$$

and this axis makes, with the direction of the field, an angle θ defined by the equation

$$\cos\theta = \alpha\lambda + \alpha'\lambda' + \alpha''\lambda'' = \frac{k\lambda^2 + k'\lambda'^2 + k''\lambda''^2}{\mathrm{H}}.$$

Denoting by M′ the moment of the couple produced by the action of the field on the sphere, we have

$$\mathrm{M}' = \phi\mathrm{M}\sin\theta,$$

or, replacing M and θ by their values,

$$M'^2 = \phi^2 M^2 (1 - \cos^2\theta) = u^2\phi^4 [H^2 - (k\lambda^2 + k'\lambda'^2 + k''\lambda''^2)^2]$$
$$= u^2\phi^4 \left[(k^2\lambda^2 + k'^2\lambda'^2 + k''^2\lambda''^2)(\lambda^2 + \lambda'^2 + \lambda''^2) - (k\lambda^2 + k'\lambda'^2 + k''\lambda''^2)^2 \right]$$
$$= u^2\phi^4 \{ [\lambda'\lambda''(k' - k'')]^2 + [\lambda''\lambda(k'' - k)]^2 + [\lambda\lambda'(k - k')]^2 \}.$$

The sphere can only be in equilibrium if the couple produced by the action of the field is null; hence the three squares comprised within the brackets must be separately null. As, by hypothesis, the co-efficients k, k' k'' are different, two of the cosines λ, λ', λ'' must be equal to zero, and therefore one of the principal axes of magnetisation must coincide with the direction of the field.

The preceding calculation applies also to the case of a homogeneous body of any given shape situated in a uniform field, for the magnetic moment, in reference to one of the principal axes, is simply proportional to the volume of the body, and to the component of the force of the field.

If the field is variable, we may suppose the volume of the body under consideration to be infinitely small; the moment of the couple which tends to turn it about its centre of gravity will still have the same expression as a function of the strength of the field at the point occupied by the element of volume.

393. EXPERIMENTAL DETERMINATION OF THE COEFFICIENTS OF MAGNETISATION.—When a cylinder is magnetised in a manner uniformly parallel with the axis, the action which it exerts on a point in the interior only depends on the two terminal layers. If the cylinder is very long, this action may be neglected for all points whose distance from one end is very great compared with the diameter; the resultant force will then be produced by external masses alone. If the external field is uniform and parallel to the axis, the magnetisation in the greater part of the cylinder will be uniform and proportional to the strength of the field. In the neighbourhood of the ends only, the induced magnetisation will be slightly modified; the superficial layer, instead of being uniform, and limited to the terminal surface, will have a more complicated distribution, and will be partially spread on the lateral surfaces.

According to this the coefficient of magnetisation of an isotropic substance may be defined as the quotient, by the force of the field, of the intensity of magnetisation which an infinitely thin cylinder of the substance acquires when placed in a uniform field; or the magnetisation which it assumes in a field equal to unity.

In like manner, the coefficient of magnetisation of an anisotropic medium in a determinate direction is the longitudinal magnetisation which an infinitely thin cylinder would acquire, parallel to this direction, in a field equal to unity.

394. We see also that to determine the coefficient of magnetisation of highly magnetic bodies such as iron, we cannot make use of the external action produced by spheres or by elongated bodies in a direction perpendicular to the field. For the ratio of the magnetisation to the force is respectively equal to

$$\frac{3}{4\pi}\cdot\frac{1}{1+\frac{3}{4\pi k}}, \qquad \frac{1}{4\pi}\cdot\frac{1}{1+\frac{1}{4\pi k}}, \qquad \frac{2}{4\pi}\cdot\frac{1}{1+\frac{2}{4\pi k}},$$

according as the body is a sphere, a disc, or a very elongated ellipsoid of revolution. These ratios differ too little from the approximate values obtained by making k infinite, to allow us to deduce the coefficient k with any degree of precision.

On the contrary, with elongated bodies parallel to the lines of force, the magnetisation tends to become proportional to k, and independent of the shape of the body.

395. Displacement of Bodies in a Magnetic Field.—Attractions and Repulsions.—The potential energy of an infinitely small dielectric sphere (178) in a field is

$$W = -\frac{u\mathrm{K}}{2}\phi^2 = -u\frac{3h}{4\pi}\frac{\phi^2}{2}.$$

This expression represents also the energy of an infinitely small magnetic sphere, and even of any volume element of a homogeneous, isotropic substance, whose coefficient of magnetisation, positive or negative, is very feeble; we have then $\frac{3h}{4\pi}=k$, and $W=-uk\frac{\phi^2}{2}$.

For a very small displacement, the variation of energy is

$$dW = -\frac{uk}{2}d(\phi^2).$$

If the body is magnetic the coefficient k is positive, the energy diminishes when the body approaches points where the absolute

value of the force is a maximum. A very small magnetic body in a variable field tends then to move towards points where the force is a maximum. As there is no absolute maximum of force outside acting masses (180), it follows that, if the body is left to itself, it will end by touching the surface of the magnets; it is therefore *attracted* by the magnets.

For diamagnetic substances the coefficient k is negative. A small diamagnetic body approaches points where the force is a minimum; it tends to move more and more away from the centres of force—it is *repelled* by magnets.

As the field may contain points where the force is null, and which are then absolute minima for the value of ϕ_2, we see that there may be stable equilibrium for a diamagnetic body in a variable field outside acting masses.

Faraday had already announced as a result of experiment this law, *that magnetic bodies move towards points where the force is a maximum, and diamagnetic bodies towards points where the force is a minimum.* It is to Sir W. Thomson that we owe the true interpretation of the phenomenon.

In a uniform field the energy of a small isotropic, magnetic, or diamagnetic body is constant, and therefore the force null.

396. For an anisotropic magnetic body the total energy is the sum of the energies corresponding to the magnetic moments $u\phi k\lambda$, $u\phi k'\lambda'$, $u\phi k''\lambda''$, due to the components $\phi\lambda$, $\phi\lambda'$, $\phi\lambda''$ of the force parallel to the three axes; we have then

$$W = -u\frac{\phi^2}{2}(k\lambda^2 + k'\lambda'^2 + k''\lambda''^2).$$

If the body is compelled to turn about its centre of gravity, stable equilibrium corresponds to the case in which the energy is a minimum—that is to say, where the expression comprised between the brackets is a maximum.

As it is necessary, for equilibrium, that two of the cosines λ, λ' and λ'' are zero, this maximum will take place when the quantity in brackets is reduced to the term corresponding to the greatest of the coefficients k, k' and k''. The axis of greatest magnetisation is then parallel to the force of the field.

If a body passes from a position in which the force and the direction of the field are defined by ϕ_1, λ_1, λ'_1, λ''_1, to another

position in which the values of the same quantities are ϕ_2, λ_2, λ'_2, λ'', the change of energy is

$$W_2 - W_1 = -\frac{u}{2}\left\{\phi_2^2[k\lambda_2^2 + k'\lambda'^2_2 + k''\lambda''^2_2[- \phi_1^2[k\lambda_1^2 + k'\lambda'^2_1 + k''\lambda''^2_1]\right\}.$$

This variation is negative, and the displacement tends to take place under the influence of magnetic forces alone, when the quantity in brackets is positive.

If the body is compelled to move parallel to itself, the preceding expression becomes

$$W_2 - W_1 = -\frac{u}{2}[k\lambda^2 + k'\lambda'^2 + k''\lambda''^2](\phi_2^2 - \phi_1^2).$$

We thus see that the body tends in all cases to move towards points in which the force is a minimum. This tendency will be the more marked, the greater is the second factor; it is a maximum when the principal axis of the greatest magnetisation, is parallel to the field, and a minimum when it is perpendicular to it.

397. If the body is diamagnetic, the values of k, k', k'' are negative, and the conclusions are just the opposite of the preceding. Stable equilibrium in a uniform field takes place when the axis of feeblest magnetisation is parallel to the direction of the field. In a variable field the body tends to move in a direction in which the force decreases, and the action is a maximum when the axis of the feeblest magnetisation is parallel to the lines of force. These two causes may act in opposite directions, and produce opposite effects according as one or the other predominates. In this way may be explained many experiments which have long appeared contradictory or paradoxical.

The results would be still more complicated for bodies whose three principal coefficients of magnetisation are not all of the same sign.

Such bodies would be magnetic in certain conditions and diamagnetic in others. None such are known; but the case might be realised artificially by placing in a non-uniform field, a crystallized magnetic sphere surrounded by an equally magnetic liquid, whose coefficient of magnetisation would be intermediate between the coefficients of the greatest and the smallest magnetisation of the sphere. The sphere would be magnetic along the axis of the

greatest, and diamagnetic along the axis of the feeblest magnetisation; it would turn in the direction of increasing forces, when the first of these axes was parallel to the field, and in the opposite direction when it is the second. These actions, however, would be so feeble that it would be difficult to make them evident.

398. Equilibrium of Long Bodies in a Uniform Field.—We have seen in electrostatics (185) that an elongated conductor placed in a uniform field is in equilibrium when the axis of the cylinder is perpendicular or parallel to the force of the field, and that the equilibrium is unstable in the first case, and stable in the second.

This must also be the case with a long iron cylinder placed in a uniform magnetic field, for the magnetisation of a soft iron sphere is a fraction very near unity, $h=\frac{\mu-1}{\mu+2}$, of the electrification which this sphere would acquire in a uniform electrical field in which the forces had the same absolute values.

It has been known in fact, since Gilbert's time, that a soft iron needle movable about a vertical axis sets in the magnetic meridian, and that if it were movable about its centre of gravity it would take up the direction of the dipping needle.

Nevertheless, in order to explain this experiment it is not sufficient to say that the magnet is everywhere magnetised parallel to the force of the field, for in that case the needle should be in equilibrium in all positions; hence the magnetisation of the mass cannot be uniform.

The couple which acts on the needle when it is oblique to the forces of the field, is due to the fact that the reactions of the various particles have modified the magnetisation; the direction parallel to the field is that therefore which, in consequence of these reactions, corresponds to the maximum of magnetisation.

399. Consider, in fact, a series of balls of soft iron B, B′, B″ fixed on a non-magnetic axis, and placed in a uniform field; let α be the angle of the axis with the direction of the field.

If the balls are so far apart as not to act on each other, the magnetisation is parallel to the field, and the resultant is null. But if the distance of the balls is not very great compared with their dimensions, it is clear that the magnetisation of each of them is increased by their mutual action, and that it takes place along directions which make, with the axis, angles ω, ω' . . . smaller than α, and changing from one sphere to another. Each sphere is no longer in equilibrium; it is under the action of a couple, and the whole of

the couples tends to bring the common axis into the direction of the field. In this position the magnetisation is a maximum.

If, on the contrary, the axis is perpendicular to the direction of the field, the reciprocal actions tend to diminish the magnetisation of each of the insulated balls; the equilibrium is unstable, and the magnetisation of the system of the spheres is minimum.

Thus the existence of a position of stable equilibrium for a magnetic needle in a uniform field, implies the existence of interactions between the different magnetic elements which constitute it, and in contradiction with Poisson's hypothesis on the constitution of magnetic bodies, which presupposes that such actions do not take place.

400. The conclusions would be almost the same for the equilibrium of a diamagnetic body, although the actions are in the opposite direction.

For the induced magnetisation is then in the opposite direction to the magnetising force. A series of balls B, B′, B″, . . ., arranged on a straight line perpendicular to the field, becomes magnetised in a direction opposite to that of the field; the reactions increase then the magnetising force on each of the balls. This direction corresponds thence to a maximum of magnetisation and to a state of equilibrium.

Let us now suppose that the line of the balls forms an angle α with the direction of the field; since the direction of magnetisation is inverse, and each pole tends to develop a pole of the same kind in the nearest part of another ball, the reactions diminish the magnetising force, and modify its direction; the effect is further the more marked the smaller the angle α. The couples which act on the spheres tend to set the axis in a direction parallel to the field. The magnetisation is then a minimum.

Hence a diamagnetic needle should also take up a direction parallel to that of the field to be in stable equilibrium.

Nevertheless the coefficient of magnetisation for diamagnetic bodies is so feeble, that the reactions of the particles may be neglected and their effect escape all means of observation.

For a diamagnetic needle, provided it is not crystalised, is in mobile equilibrium in a uniform magnetic field; in all experiments in which there seem to be phenomena of direction, the effect is due to the *magneto-crystalline* properties (397) of the body in question.

401. Equilibrium of Bodies in a Variable Field.—In a variable field the phenomena are more complicated.

Diamagnetic bodies simply follow the law of Faraday—that is to

say, that each of the volume-elements tends to move towards points where the force is a minimum, and the movement of the whole of the system is determined by this tendency of each element.

Consider, for instance, the field produced by the opposite poles of two identical magnets, or by the two poles of a horse-shoe magnet, or more simply the field of two equal masses of opposite signs (Fig. 34).

In the centre of the figure O, at an equal distance from the two magnets, the value of the force is a minimum in reference to the diametrical line AA′, and a maximum in reference to a direction Oy perpendicular to the former. A small isotropic magnetic sphere, which can only move along the right line Oy, moves towards the point O when it is in stable equilibrium; a diamagnetic sphere in the same conditions would be in unstable equilibrium at the point O, and would tend to move away to an indefinite distance. Even if this sphere were absolutely free, and situate on the right line Oy, at a small distance from O, it would move away from this point along the line Oy (that is, *perpendicularly to the lines of force*), for that is the direction in which the force varies most rapidly.

402. A long magnetic needle, movable about the point O, would set parallel to the line of the poles AA′ in stable equilibrium; each of the volume-elements would tend towards points where the force is a maximum.

A diamagnetic needle, on the contrary, would be in stable equilibrium in a direction perpendicular to the line of the poles.

The needles set then parallel, or transversely to the line joining two opposite poles, according as the coefficient of magnetisation is positive or negative. Hence the names *paramagnetic*, or *diamagnetic*, given by Faraday to bodies belonging to the first or second class.

403. We have seen that even in a uniform field, a magnetic needle places itself parallel to the lines of force, and on the other hand the different elements tend towards points where the force is a maximum.

When the two kinds of actions are concordant, as in the preceding case, the position of equilibrium can be easily determined; but it may happen that the tendency of each element to move towards the maxima of force may have the result of bringing the system into a direction which is not parallel to the lines of force. The position of equilibrium depends, in that case, on the conditions of experiment. Suppose, for instance, a series of identical soft iron needles arranged perpendicularly, and at equal distances from each other, on a non-magnetic rod, and let this system be placed between the

opposite poles of two magnets. If the needles are at considerable distances each of them will tend to put itself parallel to the lines of force, and the entire system will be in equilibrium when perpendicular to the line of the poles. If, on the contrary, the needles are gradually shortened, or if they are multiplied so that they are almost in contact, a moment will arrive in which the tendency of each to move towards points of maximum force will predominate, and the whole system will now set parallel to the lines of force—that is, to the line of the poles.

It will be seen that all intermediate cases may present themselves, and even that for a given magnetic system the direction of parallel or transverse equilibrium depends on the law of variation of the field in which it is placed.

404. Oscillations of an Infinitely Small Isotropic Needle. —The problem is identical with that which has already been treated for dielectrics (**183**, **184**).

As a particular case, if the field is symmetrical in reference to the centre of the needle, the time of the oscillations is given by the formula

$$t^2 = \pi^2 \frac{\rho}{K} \frac{1}{A+B};$$

it is independent of the length of the needle. This latter fact had been found experimentally by Matteucci for non-crystalline bismuth needles; the explanation was given by Sir W. Thomson.

In the present case, the coefficient

$$K = \frac{k}{1 + \frac{4}{3}\pi k}$$

reduces sensibly to a constant for great values of k, and becomes equal to k for small ones. The method of oscillations could not then be employed to determine the coefficient of magnetisation of highly magnetic bodies such as iron; it serves very well on the contrary for feebly magnetic or for diamagnetic bodies.

If the field varies in any way, the method of oscillations would with difficulty give good determinations of the value of k even for bodies with a very feeble coefficient. The position of equilibrium of the needle depends then, as we have seen, on the law of the variation of the field, and on the length of the needle; this is also the case with the duration of the oscillations.

405. Influence of Temperature.—Temperature has a pronounced influence on the value of the coefficient k; there is nevertheless considerable uncertainty as to the law of the variation.

The fact longest known, and best marked, is that soft iron loses almost all magnetic properties at a red heat. This is also the case with nickel at a temperature of 300 degrees; but with cobalt it only occurs near the temperature at which copper melts.

If we only take into account temperatures between -20 and 150 degrees, we find that the inducing power of iron is virtually constant, although there is reason to think that it increases at first and passes through a maximum; that that of nickel decreases continuously; and lastly, that that of cobalt constantly increases. In the case of this latter metal there is certainly a maximum towards a red heat.

Heat acts also on crystalised magnetic or diamagnetic bodies in such a way as not only to diminish the coefficients, but also to diminish the magneto-crystalline properties which are closely connected with the difference of these coefficients. In the case of bismuth the difference of the coefficients diminishes by one-half between 30 and 140 degrees; and for iron carbonate by two-thirds between the same limits of temperature.

CHAPTER V.

ON MAGNETS.

406. MAGNETISATION.—In order to magnetise a body endowed with coercive force—a steel bar, for instance—it may be placed in a constant magnetic field, or its various parts may be successively submitted to the action of a variable field, like that produced by rubbing it with a magnet.

This latter method is the oldest, and is that most frequently employed. Each point takes at every moment a magnetisation depending on that already obtained, on the actual resultant force, and, to some extent, on the time during which it acts.

Whatever method may be employed, part of the magnetism developed is *temporary*, and disappears with the external forces. Another part is *permanent* or *residual*, and all experiments show that these two kinds of magnetisation have a maximum limit.

The temporary magnetisation is greater, and the residual magnetisation less, with iron than with steel; but both have a maximum which depends only on the quality of the substance. In the case of very feeble forces magnetisation seems to be altogether temporary both for steel and for iron.

The general problem of magnetisation would consist in determining what would be the temporary magnetisation at each point of a body of given shape, and nature, subjected to known forces; and what, when these forces are suppressed, would be the residual magnetisation. This problem has only been solved theoretically in a very small number of cases.

407. INDUCTION OF A MAGNET ON ITSELF.—DEMAGNETISING FORCE.—The total magnetism of a magnet must be considered as made up of two parts, the one due to magnetic masses kept fixed by the coercive force, and which may be called *rigid magnetism*, the other resulting from the induction of the first on the magnetic body, and which constitutes *induced magnetism*.

The internal action of induced magnetism is clearly in the opposite

direction to the force which produces it; it follows that the induction of a magnet on itself always tends to diminish the magnetisation, and acts like a *demagnetising force.*

The *apparent* magnetism, or that whose effects we can observe, arises from the superposition of these two magnetisms. Hence the determination of the intensity, and of the distribution of the apparent magnetism, generally presents great difficulties.

The problem is simplified when the demagnetising force is proportional at each point to the rigid magnetism at this point; the law of distribution is then the same as if this secondary effect of induction did not take place.

In particular, when the rigid magnetism is uniform, the apparent magnetism will itself be uniform, if the secondary inductive action is constant in the interior of the magnet.

This condition is realised, as we have seen above, for a uniformly magnetised sphere; and also for an ellipsoid with a uniform magnetisation parallel to one of the axes, and for a straight unlimited circular cylinder, magnetised perpendicularly to the axis.

408. Let us first consider a sphere. Let I be the rigid, I′ the induced, and I_1 the apparent magnetisation; the demagnetising force is then (355) equal to $\frac{4}{3}\pi I_1$, and we have

$$I' = kF_1 = k\frac{4}{3}\pi I_1 = k\frac{4}{3}\pi(I - I').$$

From which we deduce

$$(1) \qquad I' = I\frac{1}{1+\frac{3}{4\pi k}} = I_1\frac{4}{3}\pi k.$$

For an ellipsoid magnetised parallel to one of the axes, the demagnetising action has the value I_1L, I_1M, or I_1N, according to the axis along which it acts (356). It is $4\pi I_1$, or $\pi^2\sqrt{1-e^2}\, I_1$ for a disc, according as it is magnetised transversely or parallel to a diameter (357). For an elongated ellipsoid of revolution, it is $2\pi I_1$ if the magnetisation is transverse, and $4\pi I_1\frac{b^2}{a^2}\left(l.\frac{2a}{b} - 1\right)$ if the magnetisation is longitudinal (357).

This latter expression tends towards zero as the ratio $\frac{b}{a}$ gradually diminishes. The demagnetising force would be still smaller for a long cylinder (373).

Hence the shape of thin plates, or of very long cylinders, is that best fitted for obtaining permanent magnets, for the demagnetising force is then the least possible. These are, in fact, the shapes which have been adopted in practice. Experiment shows, moreover, that the influence of temper is then far less than in the case of short and thick magnets. Coulomb had already observed that tempering has but a very slight influence on the magnetic rigidity of a steel wire.

409. PARTICULAR CASES OF MAGNETISATION.—*Sphere.*—It follows from the preceding discussion that a solid homogeneous and isotropic steel sphere, placed in a uniform magnetic field, will acquire a uniform temporary magnetisation, and will then retain a uniform residual magnetisation.

The expression for the temporary magnetisation will be of the form

$$(2) \qquad I = \frac{k}{1 + \frac{4}{3}\pi k} F,$$

in which the coefficient k must be regarded, not as a constant quantity, but as a function of the intensity F of the true field; the fraction by which the force F must be multiplied to get the magnetisation I, tends in fact to become inversely as F—that is to say, equal to $\frac{I_0}{F}$, as F increases, since the magnetisation tends towards a maximum I_0.

In like manner, the residual magnetisation is a fraction of the temporary magnetisation; a variable fraction, and one which tends towards a limiting value $\frac{1}{m}$, since the residual magnetisation has a maximum, and is then $\frac{I_0}{m}$, a fraction of the maximum temporary magnetisation.

In all cases the law of distribution is the same; the density at every point is equal to the perpendicular projection of the magnetisation—that is to say, proportional to the abscissa of the point measured from the centre along the diameter, parallel to the magnetisation. The linear density measured along the same axis is also proportional to the abscissa. The moment of the sphere is uI_1, the total mass of each of the layers $\frac{3}{4}\frac{uI_1}{a}$, the distance of the poles is $\frac{4a}{3}$, and each pole is $\frac{2}{3}$ of the radius from the centre.

410. *Ellipsoid.*—This is also the case with a homogeneous and isotropic ellipsoid, one of whose axes coincided with the direction of the uniform field during magnetisation; it is merely necessary to replace the factor $\frac{4}{3}\pi$ by a coefficient L which depends on the form of the ellipsoid (357).

The maximum magnetisation I_0, and the fraction $\frac{I}{m}$ which determines the maximum residual magnetisation, have values which are connected with those which correspond to the sphere by ratios depending on the form of the ellipsoid.

The law of distribution is still known, and the poles are at a distance from the centre equal to $\frac{2}{3}$ of the semi-axis parallel to the magnetisation.

We might in like manner obtain a uniform magnet with a circular disc magnetised perpendicularly to a plane, or parallel to a diameter (357).

411. *Anchor Ring.*—A simple case, which can be easily realised experimentally, is that of a body bounded by a closed tube—a torus or anchor ring, for instance—in which the magnetisation would be everywhere parallel to the axis. The magnet may then be regarded as formed of simple solenoids closed on themselves (371). The external action of the system is always exactly null.

412. *Cylinder.*—To the preceding examples, all of which represent finite volumes, which can be exactly realised, we may add that of a homogeneous and isotropic unlimited circular cylinder placed in a uniform field perpendicular to the axis; the magnetisation is then represented by the expression

$$I = \frac{k}{1 + 2\pi k} F.$$

These cases seem to be the only ones in which the distribution of magnetism can be theoretically determined, at least when the coefficient k is not independent of the magnetising force.

413. Any given Magnets.—Experimental Methods.—The problem of magnetisation for a body of any given form can only be attacked experimentally by the study of its external actions; we have already had occasion to remark that our knowledge of the external field of a magnet can teach us nothing about the internal distribution of magnetism; it only enables us to determine the distribution of the fictive layer, equivalent to the real magnetisation.

We shall mention here the principal experimental methods used, in order to define their theoretical meaning.

414. *Oscillations.*—This method, which was used by Coulomb, consists in making a very small horizontal needle oscillate in front of several points of a bar placed vertically in the meridian plane of the needle. If n and N are the number of oscillations made by the needle under the sole action of the earth, and under the combined action of the earth and of the bar, the action of the bar on the needle (the magnetism of which is supposed unchanged) is proportional to the difference $N^2 - n^2$ of the squares of the two numbers. We measure thus the perpendicular component of the magnetic force at the point in question. Coulomb assumed that this perpendicular component was proportional to the density of the superficial fictive layer at the nearest point of the needle, except quite close to the end; in this case he determined the density either by a graphical method, or by doubling the value obtained for the oscillations of the needle. It cannot be concealed that this mode of correction is somewhat arbitrary; it is, moreover, quite inexact, as we shall afterwards see (419) that the perpendicular component at a point is proportional to the density of the corresponding fictive layer, and that it may directly give the distribution of magnetism.

415. *Torsion Balance.*—A second method, also due to Coulomb, consists in measuring the repulsion exerted by every point of the magnet, at a constant and very small distance, on the pole of a long needle movable in a plane perpendicular to the axis of the bar. If we regard the pole of the needle as unchanged, the torsion imparted to the wire by which it is suspended to keep the needle in the desired position, measures the perpendicular component of the magnetic force with a certain degree of approximation.

416. *Use of Soft Iron.* — In the two preceding cases it is assumed that the magnetism of the auxiliary magnet is invariable, so that the action which it experiences is simply proportional to the strength of the field. If the oscillating needle is of soft iron, and the magnetisation of this needle is proportional to the strength of the field, the action which it undergoes will be proportional to the square of the perpendicular component.

We may, in like manner, place a piece of soft iron (M. Jamin's *test nail*) in different parts of the magnet, and determine the force necessary to detach it; this force of tearing away is still within certain limits proportional to the square of the normal component.

In these two methods, however, we do not take into consideration either the variation of the coefficient k with the strength of the

magnetic force, nor of the modifications produced by the presence of soft iron in the magnetic state of the bar exactly in the region we are exploring. The results furnished by the use of soft iron do not then seem to be so definite as those obtained with magnets.

417. *Measurement of the Flow by Induction Currents.* — This method is the only one which gives exact results; the theory will be given subsequently. It is sufficient here to remark, that by means of induction currents we may determine the flow of force, or the flow of magnetic induction across a closed circuit.

If the bar is anywhere surrounded by a ring formed of one or many turns and connected with a galvanometer, and if by any method we suddenly suppress the magnetisation, the momentary current produced in the ring measures the total flow of induction which traverses the plane bounded by the ring at the point in question; if the ring clasps the bar tightly, the flow of induction which traverses the ring is that which exists in the section of the bar itself.

The ring being placed in the same point it is caused to glide along the axis of the bar to a distance which may be regarded as infinite; the current, in this case, measures the total flow of force emanating from the magnet, measured from the point of departure.

Experiment shows, as indeed is evident from the theorem of the conservation of the flow of induction, that the current is the same as in the preceding case.

By measuring in either way the flow corresponding to different points, we may construct a curve which represents the magnetic condition of the bar. The curve has a maximum ordinate which corresponds to the neutral line; it sinks on each side and becomes an asymptote to the axis of the bar, which we suppose to be prolonged indefinitely. This may be called with Gaugain the *curve of demagnetisation.*

If, while the ring is at a point, the abscissa of which is x, it is displaced by a quantity dx, the current measures the external flow corresponding to this length dx, or, what is the same thing, the variation in the internal flow of induction. By successively displacing the ring by equal amounts, we may construct the curve whose ordinates represent the external flow, and therefore the perpendicular component at the various points. The ordinates of this curve are the differentials of the ordinates of the curve of demagnetisation.

This method furnishes then, like the preceding, but in an exact manner, the values of the perpendicular component at every point of the bar.

418. DISTRIBUTION OF THE FICTIVE LAYER.—The fictive layer is not a layer of equilibrium, but we know that its density at each point (39) satisfies the ratio

$$\sigma = \frac{1}{4\pi}(F_n - F'_n),$$

in which F_n and F'_n denote for two infinitely near points on each side of the surface, the first outside and the second inside, the perpendicular components of the actions exerted by the external masses, and by the layer. The preceding methods give the component F_n, but the component F'_n is in general unknown; hence they only enable us to determine the density of the fictive layer in certain special cases.

It may happen, in fact, that the fictive layer may replace magnetic masses which really exist in the magnet not only for external, but also for internal points; and this is what takes place in the phenomena of magnetic induction, when the coefficient k is constant. There is then a constant ratio μ between the external and internal perpendicular components, and the expression for the density is

$$\sigma = \frac{1}{4\pi}\frac{\mu - 1}{\mu} F_n = \frac{k}{1 + 4\pi k} F_n.$$

In this case the distribution is completely known when we know the external perpendicular component at every point. This is not the case if the coefficient k is variable, and still less if there is rigid magnetism.

The ordinary methods do not give directly the distribution of the fictive layer in a magnetised bar; it is incorrect, in particular, to consider the abscissa of the centre of gravity of the curve of the perpendicular components as giving the position of the pole. This is readily seen, if we examine the case of a cylinder magnetised uniformly in a direction parallel to the axis. We have seen (373) that its action may be represented by that of two layers, one negative and the other positive, distributed uniformly on each of the bases. It is easy to see that the flow of force for the lateral surface is not zero, although the density is zero. The centre of gravity of the curve representing the flow across the lateral surface is in the interior of the magnet, while this pole is exactly situate on the terminal surface.

We may observe that if the perpendicular components do not give the distribution, they enable us to calculate the total mass of magnetism by Green's theorem. This mass is obviously null for the whole magnet; but the total flow of force on either side of the neutral line is equal to 4π by the mass of the fictive layer corresponding to this side. This total mass is represented by the area of the curve obtained by taking as ordinates the values found for the perpendicular component at all points of the axis of the magnet, imagined to be indefinitely prolonged, or more simply by the maximum ordinate of the curve of demagnetisation.

419. CYLINDRICAL MAGNETS.—Coulomb determined experimentally, and by means of the methods mentioned above, what he calls the distribution of magnetism in cylindrical needles.

He first found that for *short* magnets—those, that is to say, whose length is less than fifty times the diameter—the perpendicular force at each point (which he confounded with the density) is proportional to the distance from the middle.

The linear density would then be the same as for a sphere or an ellipsoid uniformly magnetised.

The curve of distribution is then figured by a right line OB (Fig. 86), making a certain angle α with the axis OA of the bar.

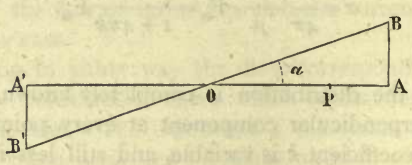

Fig. 86.

A straight line OB', forming the prolongation of the former, would represent the negative magnetism on the other half of the bar. The centre of gravity of the surface is projected, as for a sphere, at a third of the semi-length of the bar measured from the ends.

This law ought to represent the true distribution of magnetisation with tolerable approximation, for Coulomb proved that, other things being equal, the magnetic moment of short bars is proportional to the cube of the length.

If the bar is *long*—that is to say, if the length is more than fifty times the diameter *d*—the magnetism is imperceptible for a certain length on either side of the centre, and may still be represented by a

triangle CAA′ (Fig. 87), the base of which is equal to twenty-five times the diameter. The angle α of the right line which represents the densities is constant for bars which only differ in length. The quantity of magnetism is then constant, and is the same as in a limited magnet, for which we should have $L = 50d$; this quantity may then be represented by $a\,(50d)^2$ and the moment by $a\left(L - \frac{50}{3}d\right)(50d)^2$.

Coulomb, however, only considered these results as a first approximation. He observed that if we take equidistant parts from the end A of a magnet, the successive tangents to corresponding points of the curve make with each other equal angles. The curve which satisfies this condition is given by the equation $e^{-\beta y} = \cos \beta x$, which for small values of x merges into an arc of a parabola CB (Fig. 87) tangential to the axis at a point C, at a distance l, from the end; the quantity of magnetism is then proportional to l^3, that is bl^3, and the pole is at a distance from the end equal to $\frac{l}{4}$. The magnetic moment has the value $\left(L - \frac{l}{2}\right)bl^3$.

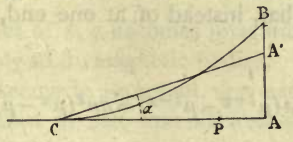

Fig. 87.

It will be seen that the magnetic moment for a very long cylinder tends to become proportional to the length, as in the case of induced magnetisation.

420. Empirical Formulæ.—These two portions of a parabola do not represent the distribution of magnetism by a continuous function. Biot found that Coulomb's experiments are represented very exactly by the exponential formula

$$y = a(\mu^x - \mu^{2l-x}), \tag{3}$$

in which y is the magnetism at a point at a distance x from one end, a and μ are constants.

Biot arrived at this formula by comparing the magnet to a Volta's pile, which he considered as being itself a series of plates in which the electricities of the terminal plates A and B *dissimulate* quantities of electricity of opposite signs which vary in geometrical progression with the number of plates. If N be the total number of plates, the positive electricity of A dissimulates in the n^{th} plate a quantity of negative electricity expressed by $a\alpha^n$, and the negative electricity of B dissimulates, in the same element, a quantity of positive electricity $a\alpha^{N-n}$, so that the quantity of free electricity in this element, supposing the terminal charges to be equal, is

$$y = a(\alpha^n - \alpha^{N-n}).$$

We may get the previous formula from this by putting $N = 2lp$, and therefore $n = xp$, p being the number of pairs for unit length, and taking $\mu = \alpha^p$,

It seems difficult to discuss a mode of reasoning which has for its basis only the vague notion of dissimulated electricity.

It may be observed that if we take the origin of the co-ordinates in the centre of the bar, instead of at one end, equation (3) may be written

$$z = a(\mu^{l+x} - \mu^{l-x}) = a\mu^l(\mu^x - \mu^{-x}).$$

421. Green, starting from a particular conception of the coercive force, found that, for a circular cylinder placed in a uniform field parallel to the axis, the linear density at a distance x from the middle of a bar whose length is $2l$, and radius a, might be expressed by the formula

$$\lambda = \pi k \mathrm{F} q a \frac{e^{\frac{q}{a}x} - e^{-\frac{q}{a}x}}{e^{\frac{q}{a}l} + e^{-\frac{q}{a}l}}, \tag{4}$$

or putting $\frac{q}{a} = \beta$,

$$\lambda = nk\mathrm{F}qa \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta l} + e^{-\beta l}},$$

in which F represents the strength of the field, and q a constant given by the equation

$$0.231863 - 2\,l.q + 2\,q = \frac{1}{\pi k q^2}.$$

Green assumes that the coefficient of magnetisation k, is constant throughout the whole extent of the body; in this case the linear density is proportional to the perpendicular component of the magnetic force at every point of the surface. Maxwell gives the following table of the corresponding values of q and of k:—

k	q	k	q
∞	0.00	11.80	0.07
336.4	0.01	9.13	0.08
62.02	0.02	7.52	0.09
48.41	0.03	6.32	0.10
29.47	0.04	0.143	1.00
20.18	0.05	0.0002	10.00
14.79	0.06	0.0000	∞

For negative values of k, q becomes imaginary; the formula does not seem then to apply to diamagnetic bodies.

Green's formula seems to represent very exactly the distribution of temporary magnetism in soft iron as well as that of permanent magnetism in cylindrical bars. Green showed that the value of the moment which is deduced for a needle of this form,

$$(5) \qquad m = \pi a^2 k \mathrm{F} \left(2l - \frac{2}{\beta} \frac{e^{\beta l} - e^{\beta l}}{e^{\beta l} + e^{\beta l}} \right),$$

agrees remarkably with determinations made by Coulomb with needles which only differed in length. The agreement ceases, however, to be very close when the length of the needle is less than twenty-five times the diameter.

The expression for the area of the curve corresponding to Green's formula is, for each half of the bar,

$$\mathrm{S} = \pi a^2 \mathrm{F} k \left(1 - \frac{2}{e^{\beta l} + e^{-\beta l}} \right);$$

it represents the total value of the flow of lateral force. For a very

long cylinder it reduces sensibly to $\pi a^2 F k$; the flow from the ends may then be neglected.

If we assume that the abscissa of the centre of gravity of this area determines the position of the pole, we shall obtain the distance $2d$ of the two poles by dividing the moment m by the mass S; we shall thus obtain

$$(6) \qquad 2d = \frac{2l - \dfrac{2}{\beta}\dfrac{e^{\beta l} - e^{-\beta l}}{e^{\beta l} + e^{-\beta l}}}{1 - \dfrac{2}{e^{\beta l} + e^{-\beta l}}}.$$

for very long needles, this expression reduces to $2\left(l - \frac{1}{\beta}\right) = 2\left(l - \frac{a}{q}\right)$, that is to say, that the poles are at a distance $\frac{a}{q}$ from the ends.

422. M. Jamin obtained an analogous expression. If y is the *tension* at each point, or the density, p and s the perimeter and section of the bar, and A and c two constants, M. Jamin, in comparing the phenomenon to the propagation of heat, finds, by Fourier's laws, the following formula, which agrees with the results of his experiments:—

$$y = \mathrm{A}c\sqrt{\frac{s}{p}}e^{-cl\sqrt{\frac{p}{s}}}\left(1 - e^{-cl\sqrt{\frac{2p}{s}}}\right)\left[e^{-cx\sqrt{\frac{p}{s}}} - e^{-cx\sqrt{\frac{p}{s}}}\right].$$

If the section of the bar is circular and of radius a, we have $\sqrt{\frac{p}{s}} = \sqrt{\frac{2}{a}}$, and putting $c\sqrt{2} = \mathrm{B}$, the formula becomes

$$(7) \qquad y = \frac{\mathrm{AB}}{2}\sqrt{a}\,e^{-\frac{\mathrm{B}}{\sqrt{a}}l}\left(1 - e^{-\frac{\mathrm{B}}{\sqrt{a}}2l}\right)\left(e^{\frac{\mathrm{B}}{\sqrt{a}}x} - e^{-\frac{\mathrm{B}}{\sqrt{a}}x}\right);$$

it becomes identical with that of Green if we put

$$\beta = \frac{q}{\sqrt{a}} \qquad \text{and} \qquad \mathrm{A} = \frac{2\pi a k \mathrm{F}}{1 - e^{-4\frac{q}{a}l}}.$$

For long bars, this latter condition reduces to

$$A = 2\pi ak F = pkF,$$

and shows that the constant A is proportional to the perimeter.

423. Green's formula corresponds to the case of a cylinder placed in a uniform field parallel to the axis, and for which the coefficient of magnetisation is constant. Professor Rowland has pointed out the analogy of this formula with that which expresses the lateral flow from a pile of the same form placed in a conducting medium (**268**). Let us suppose that the flow of magnetic induction is propagated like the flow of electricity; if we retain the same meanings for the quantities ρ, ρ', and R_1, and if we replace the quantity ϵ by the force F of the field, and if we call Q the flow of magnetic induction across a section of the bar at the distance x from the centre, we have, for the flow in the interior,

$$(8) \qquad Q = \frac{F}{\rho}\left[1 - \frac{\beta\left(e^{\beta x} + e^{-\beta x}\right)}{\left(\frac{\rho}{R_1} + \beta\right)e^{\frac{\beta l}{2}} - \left(\frac{\rho}{R_1} - \beta\right)e^{-\frac{\beta l}{2}}}\right],$$

and, for the lateral flow,

$$\frac{dQ}{dx} = \frac{F}{\rho}\,\frac{e^{\beta x} - e^{-\beta x}}{\left(\frac{\rho}{R_1} + \beta\right)e^{\frac{\beta l}{2}} - \left(\frac{\rho}{R_1} - \beta\right)e^{-\frac{\beta l}{2}}}.$$

These formulæ also apply to the case in which the magnet is solenoidal, bounded by a channel surface closed upon itself, and the magnetisation of which is everywhere perpendicular to the right section. We have, in that case, $R_1 = 0$, $Q = \frac{F}{\rho}$, and the flow of lateral induction is zero. If F_1 is the magnetic induction, s the section of the bar, and μ the coefficient of permeability, we have, further,

$$Q = sF_1 = s\mu F,$$

and therefore $\mu = \frac{1}{\rho s}$.

424. All experiments go to prove that magnetisation tends

towards a maximum when the magnetising force increases without limit. If the values of this force be taken as abscissæ, and the values of the magnetisation as ordinates, we obtain a curve like OBL (Fig. 90), having an asymptote parallel to the axis of the abscissa, and with a point of inflexion near the origin.

Professor Rowland represents the phenomena in a different way. Taking the values of induction F_1, as given directly by experiment, as abscissæ, and the values of μ as ordinates, he finds that his experiments are represented very closely by the formula

$$\mu = a \sin \frac{F_1 + b\mu + c}{d}, \tag{9}$$

in which a, b, c, and d are constants depending on the nature and quality of the metal.

The curve represented by this equation has the general form of a parabola with a diameter conjugate with the axis of the abscissæ; it cuts this axis in two points, and the inclination of the diameter depends on the constant b. The position of the points of intersection with the axis depends on the values of c and d. The constant a evidently represents the maximum value of μ. Professor Rowland's experiments give for this maximum at the ordinary temperatures numbers between 3000 and 5000 in the case of iron, and 300 in that of nickel. The curve assumes another form when the temperature changes, and the deformation appears to be far greater for nickel than for iron.

425. HYPOTHESIS ON THE CONSTITUTION OF MAGNETS.—According to Poisson's theory the magnetisation of a medium is produced by the separation of the magnetic fluids in the interior of each particle, and as no limit can be assigned to the quantity of neutral fluid which can exist in a definite volume, the magnetisation itself might increase without a limit.

We shall afterwards see how Ampère, starting from the magnetic properties of electrical currents, was led to assume that each particle of a magnetic substance is surrounded in the natural state by an infinitely small electrical current, and constitutes an elementary magnet. In a magnetic body withdrawn from all external force, these elementary magnets are only subjected to their mutual actions, and are turned indifferently in all directions. If the body is submitted to the action of a magnetic field the axes of the different magnetised particles tend to take the direction of the field at each

point, and the magnetisation which results therefrom is the stronger, the more these particles are deviated from their original direction. If the axes of all these particles were parallel the magnetisation would have reached its maximum value.

But this position can never be attained, owing to the reciprocal reactions of the molecules. Wilhelm Weber has shown how these reactions may be allowed for.

426. WEBER'S THEORY.—Let us assume with Weber that each unit of volume contains n magnetic molecules, and that the moment of each of them is equal to m. If all these molecules were parallel, the magnetic moment of unit volume would be $M = nm$, and the magnetisation of the medium would be at its maximum.

When the medium is in the natural state, the molecules are turned indifferently in all directions. To express this property let us draw through the centre of the sphere a radius parallel to each of the axes of the n molecules; the extremities of these radii will be arranged on the sphere in a uniform manner.

The number of molecules the axes of which make, with a determinate direction, which we take for the axis of x, an angle smaller than α is $\frac{n}{2}(1 - \cos \alpha)$; and the number of molecules whose angles with the axis of x are between α and $\alpha + d\alpha$ is equal to $\frac{n}{2} \sin \alpha d\alpha$.

Let us now suppose this medium to be in a uniform field whose intensity X is parallel to the axis of x, and consider the action which it exerts on a molecule whose magnetic axis makes an angle α with the direction of the field.

If this molecule were free it would set parallel to the force of the field, and, all the other molecules undergoing an analogous rotation, the medium would attain the maximum magnetisation under the influence of any external force however feeble.

As this is not the case, it must be assumed that each molecule is impelled to resume its original direction by an antagonistic force, which arises either from the constitution of the medium itself, or by the reactions which the magnetised molecules exert upon each other.

The simplest hypothesis is to suppose that this antagonistic force D is constant, and acts in the original direction of the axis of each molecule.

The new direction of the axis of a molecule in its position of equilibrium is then given by that of the resultant of the forces D and X.

427. To get the direction of the molecule let us draw a sphere whose radius is equal to the reaction of the medium, and take a length OS from the centre, equal and opposite to the strength of the field (Fig. 88).

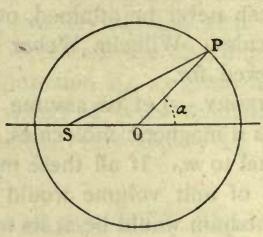

Fig. 88.

A molecule, the axis of which was originally directed along the line OP, is subject to two forces—SO and OP, the resultant of which is SP. If the resultant S is in the interior of the sphere—that is, if the reaction of the medium is greater than the strength of the field —the axes of the deviated molecules will be still in any direction whatever, but no longer uniformly.

If the force of the field is greater than the reaction of the medium, the point S is beyond the sphere (Fig. 89), and the axes of the deviated molecules are all comprised within the cone TST' tangential to the sphere.

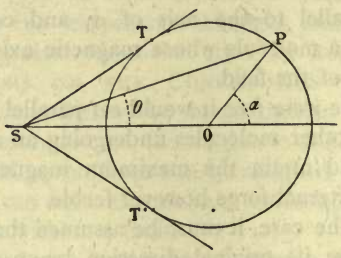

Fig. 89.

Let α be the original inclination of the axis of a molecule to the axis x,

θ the final inclination,

β the deflection $\alpha - \theta$,

R the resultant of the magnetising force X, and of the reaction D of the field.

The condition of equilibrium is

$$mX \sin\theta = mD \sin\beta = mD \sin(\alpha - \theta);$$

from which we deduce

$$(11) \qquad \tan\theta = \frac{D \sin\alpha}{X + D\cos\alpha}.$$

428. The structure of the medium being symmetrical in reference to the axis of x, the strength of magnetisation is given by the sum of the projections of the magnetic moments of all the molecules on the axis of x.

The projection of the moment of a molecule is expressed by $m\cos\theta$; the number of those which originally made the angle α with the axis of x, is $\frac{n}{2}\sin\alpha\, d\alpha$: the resultant is then

$$I = \int_0^{\pi} m\cos\theta \frac{n}{2}\sin\alpha\, d\alpha = -\int_{\pi}^{0} \frac{mn}{2}\cos\theta\sin\alpha\, d\alpha,$$

or

$$I = -\frac{M}{2}\int_{\pi}^{0} \cos\theta\sin\alpha\, d\alpha.$$

The triangle SOP gives the equation

$$R^2 = D^2 + X^2 + 2DX\cos\alpha,$$

from which is deduced

$$R\,dR = -DX\sin\alpha\, d\alpha.$$

We have further

$$D^2 = R^2 + X^2 - 2RX\cos\theta.$$

Expressing in this way the angles α and θ by their values as a function of R, we get

$$I = \frac{M}{4}\int \frac{R^2 + X^2 - D^2}{X^2 D}\, dR = \frac{M}{12DX^2}\Big[R(R^2 + 3X^2 - 3D^2)\Big]_{R_1}^{R_2}.$$

In the first case, in which we have $X<D$, the limits of the integration are $R_2=D+X$ and $R_1=D-X$.

In the second case, in which we have $X>D$, the limits of the integration are $R_2=X+D$ and $R_1=X-D$.

All reductions being made, we get then:

$$\text{When } X<D, \quad I=\frac{2}{3}M\frac{X}{D};$$

$$\text{„ } X=D, \quad I=\frac{2}{3}M;$$

$$\text{„ } X>D, \quad I=M\left[1-\frac{1}{3}\frac{D^2}{X^2}\right];$$

$$\text{„ } X=\infty, \quad I=M.$$

According to this theory, the magnetisation is at first proportional to the magnetic force until it is equal to the reaction of the medium, in which case the magnetisation attains two-thirds of its maximum value. When the magnetic force has become greater, the magnetisation increases less rapidly, and tends towards a finite limit.

The curve OL (Fig. 90), which represents the magnetisation as a function of magnetic force, consists then of a rectilinear part OA which is prolonged by the curve AL, the asymptote to a horizontal straight line CD.

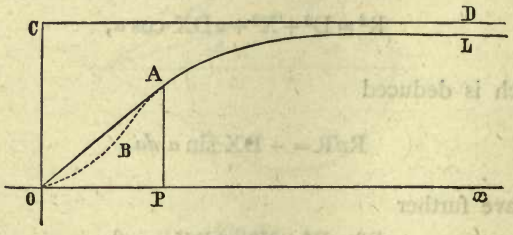

Fig. 90.

429. Weber's own experiments agree satisfactorily with this law. More recent researches, however, have shown that the value of k cannot be considered constant even for small forces. This coefficient at first increases regularly up to a maximum, and then diminishes.

The magnetisation of iron, as a function of the field, must therefore be represented by a curve such as OBA (Fig. 90), having a point of inflexion; this first part of the curve has often been confounded with the tangent passing through the origin, and which gives the maximum value for k. Weber's theory does not account for this variation of the coefficient of magnetisation for small forces; nor, on the other hand, does it throw any light on the nature of residual magnetisation.

430. MAXWELL'S THEORY.—In order to complete this last link, while still adhering to the general theory, Maxwell supposed that the medium had a kind of imperfect elasticity. He assumes that the axis of the magnetic molecules revert to their original position, after the suppression of the magnetising force, so long as the rotation which they experience is below a certain value, but that their axes retain a permanent deviation $\beta - \beta_0$, when the rotation β has been greater than the limiting value β_0. This deviation $\beta - \beta_0$ characterises the permanent condition of the molecule.

This hypothesis undoubtedly does not represent the exact state of the phenomena, but it may furnish an approximate idea, and enable us to submit the problem to calculation.

According to Maxwell, we may deduce the temporary magnetisation I and the permanent magnetisation I′ by a calculation analogous to the preceding. Putting

$$L = D \sin \beta,$$

we get thus:

When $X < L$,

$$I = \frac{2}{3} M \frac{X}{D}, \qquad I' = 0;$$

When $X = L$,

$$I = \frac{2}{3} M \frac{L}{D}, \qquad I' = 0;$$

For $L < X < D$,

$$I = M \left\{ \frac{2}{3} \frac{X}{D} + \left(1 - \frac{L^2}{X^2} \right) \left[\sqrt{1 - \frac{L^2}{D^2}} - \frac{2}{3} \sqrt{\frac{X^2}{D^2} - \frac{L^2}{D^2}} \right] \right\},$$

$$I' = M \left(1 - \frac{L^2}{D^2} \right) \left(1 - \frac{L^2}{X^2} \right);$$

For $X = D$,

$$L = M\left[\frac{2}{3} + \frac{1}{3}\left(1 - \frac{L^2}{D^2}\right)^{\frac{3}{2}}\right], \qquad I' = M\left(1 - \frac{L^2}{D^2}\right)^2;$$

For $X > D$,

$$I = M\left[\frac{1}{3}\frac{X}{D} + \frac{1}{2} - \frac{1}{6}\frac{D}{X} + \frac{(D^2 - L^2)^{\frac{3}{2}}}{6X^2D} - \frac{\sqrt{X^2 - L^2}}{6X^2D}(2X^2 - 3XD + L^2)\right],$$

$$I' = \frac{M}{4}\left[1 - \frac{L^2}{XD} + \sqrt{1 - \frac{L^2}{D^2}}\sqrt{1 - \frac{L^2}{X^2}}\right]^2;$$

Lastly, for $X = \infty$,

$$I = M, \qquad I' = \frac{M}{4}\left(1 + \sqrt{1 - \frac{L^2}{D^2}}\right)^2.$$

Fig. 91 represents the course of the phenomenon for particular values: $M = 1000$, $L = 3$, $D = 5$. The magnetising forces are taken

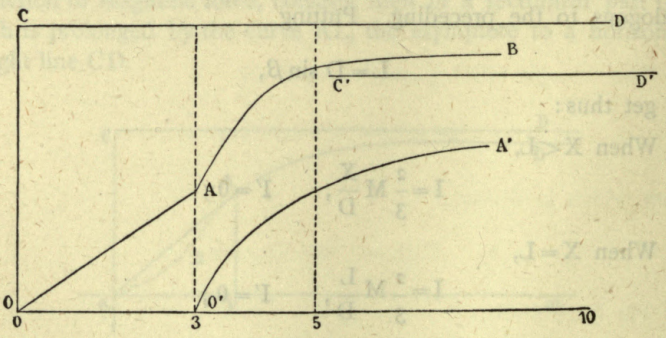

Fig. 91.

as abscissæ; the ordinates of the curve OAB represent the temporary magnetism, and that of the curve O′A′ the residual magnetism. The former consists at first of a rectilinear portion corresponding to the values of X comprised between 0 and 3; it then rises suddenly, and rapidly approaches its asymptote. The curve of residual magnetism

only commences when X = L; the maximum M' towards which it tends, and which is figured by the right line C'D', is equal to 0·81 M.

It may be remarked that the residual magnetism thus calculated, corresponds to the case in which the magnetisation of the body itself produces only an inappreciable demagnetising force; these results correspond then only to the case of a very long body magnetised longitudinally.

It is difficult to admit that a discontinuous curve like that which represents the temporary magnetism can be an exact expression of the phenomenon. Nevertheless, this theory leads to curious consequences relative to the successive action of magnetising forces of opposite signs, and which have been verified experimentally.

Let us suppose that a piece of iron after having been submitted to the action of a force X_0, has acquired a permanent magnetisation. A new force $-X_2$ of the same direction is without effect as long as it is less than X_0, and if it is greater than X_0 the residual magnetism is the same as if the original force had not acted.

If the new force $-X_2$ is in the opposite direction, it produces a permanent effect long before it reaches X_0; the residual magnetism seems to be destroyed for a certain value of this force, but the metal is not in the neutral state, for it is insensible to the action of a force $-X$, so long as X is less than X_2, while a feebler positive force produces a permanent magnetisation in the original direction.

431. JAMIN'S OBSERVATIONS.—Jamin gives a different explanation of these phenomena. He assumes that the action of the field on a bar extends to a greater or less depth according to its strength. When the apparent magnetisation has become zero, the magnetism is not destroyed; it was merely a case of the superposition of two opposite magnetisations. An inverse field of less strength than X_2 has no action on the superficial layer, but a direct field of less strength forms a new superficial magnetisation, the action of which is added to that which was previously there.

M. Jamin verified these theoretical ideas by removing the superficial layer of inverse magnetisation, and exposing the subjacent layer of direct magnetisation. He succeeded in doing this either mechanically, by grinding or filing away the outer surface of the magnet, or chemically by dissolving it with acid.

It must, however, be observed that this predominance of the surface layers is perhaps an accidental phenomenon peculiar to steel, and simply dependent on the constitution of this metal. For, in the case of highly tempered bars, such as those which are sought for the construction of magnets, the tempering is necessarily very

unequal; it is more particularly produced near the surface, where the cooling is very rapid, so that the maximum action of the coercive force is in the superficial layers. The inductive action and the demagnetising force, manifest themselves then in conditions quite different from those met with in homogeneous bodies.

432. Influence of Temperature.—The magnetism induced by the action of a magnet on itself, is perhaps the simplest way of explaining the influence of temperature.

It is natural to assume that rigid magnetism is not altered by small changes of temperature, for the magnetisation resumes its original value when the magnet regains its original temperature; it is difficult to suppose that rigid magnetism can repair its losses, for all the internal actions tend to diminish it. On these considerations, the temporary enfeeblement of magnetism will be simply due to an increase in the induced magnetism, and therefore the coefficient of magnetisation must at first increase with the temperature.

For higher temperatures, above 100° for instance, magnetism undergoes a distinct diminution; the rigid magnetism itself has therefore changed. In these conditions we cannot say whether the coefficient of magnetism continues to increase with the temperature, for the enfeeblement is due to a double cause. As iron and steel at a red heat are no longer attracted by a magnet, we must assume that the coefficient of magnetisation then becomes null, or at any rate is extremely small.

It appears then that for iron and steel the coefficient of magnetisation must at first increase with the temperature and then diminish to zero, passing through a maximum at a definite temperature.

If this is the case, a bar magnetised at a lower temperature than the maximum must lose magnetism when it is heated, and the converse must take place with a bar magnetised at a lower temperature than that of the maximum.

Experiment shows that this is the case with cobalt. For iron and steel the facts hitherto known agree partially with this mode of view; but there are too few experiments made under well defined conditions, to enable us to judge how far it agrees with the truth. Everything seems to indicate that the true phenomena are more complex.

CHAPTER VI.

MAGNETIC CONDITION OF THE GLOBE.

433. GAUSS' METHOD.—The representation of terrestrial magnetism by the hypothesis of a central magnet, or by equivalent hypotheses, only constitutes a somewhat rough first approximation; the problem is really far less simple. Gauss treated it in a completely general manner, on the hypothesis that the effects observed on the surface of the Earth are due solely to the action of magnetic masses.

Whatever may be the distribution of these masses, whether they are in the inside or on the surface of the globe, the elementary actions are exerted inversely as the square of the distance, and the force at each point is still determined by a potential. The space surrounding the earth forms the magnetic field of the system, and we may suppose it divided into layers by equipotential surfaces, corresponding to equidistant values of the potential. The surface, which corresponds to a given value V, may be formed of one or more sheets; but we know that two surfaces of different potentials do not intersect, and that the force perpendicular at each point is inversely as the distance of two consecutive surfaces.

434. MAGNETIC PARALLELS.—A certain number of these surfaces cut the terrestrial globe: *magnetic parallels* are the lines of intersection corresponding with the surface of the Earth; these lines are level lines. As they belong both to the surface of the Earth, which we suppose is spherical, and to the equipotential surface, they are perpendicular at each point to the vertical and to the magnetic force; they are therefore perpendicular to the magnetic meridian passing through these two lines, and therefore to the intersection of this meridian with the surface of the Earth—that is, to the magnetic meridian. The magnetic parallels form, therefore, on the surface of the terrestrial sphere, a system orthogonal to the magnetic meridians.

Let us consider the parallels corresponding to two infinitely near equipotential surfaces V_1 and V_2 (Fig. 92); let ds be the arc of the

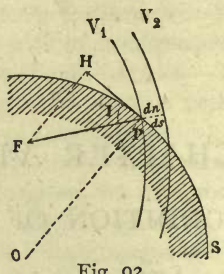

Fig. 92.

meridian comprised between them, and dn the perpendicular distance of the two surfaces at the same point. If F be the magnetic force and I the inclination, we have evidently

$$F = -\frac{dV}{dn} = -\frac{dV}{ds \cos I};$$

from which is deduced

$$F \cos I = H = -\frac{\partial V}{\partial s}. \tag{1}$$

The horizontal component, perpendicular at every point to the magnetic parallel, is therefore inversely as the distance of two consecutive parallels; but the total force and the horizontal component are not necessarily constant along a magnetic parallel, as is the case on Biot's theory.

435. MAGNETIC EQUATOR.—The sum of the magnetic masses being null for the whole system, and also separately for each of the magnetised bodies, there is a level surface for which $V = 0$; this surface cuts the terrestrial globe along its neutral line if it is the only magnetic body, or in the vicinity if the other magnetic bodies are sufficiently distant.

The magnetic parallel where the potential is zero is called the *magnetic equator;* along this equator the force is not constant, nor is it necessarily horizontal. On Biot's theory the equator was a line of which the inclination was null.

The magnetic equator separates those points on the earth for which the potential is positive, from those where it is negative. On either side of the equator the absolute value of the potential decreases continuously.

436. Terrestrial Magnetic Poles. — The term terrestrial magnetic poles, is ordinarily applied to those points of the surface where the potential is a maximum or minimum. A pole is a point where the level surface becomes a tangent to the surface of the Earth; the force there is evidently vertical.

The number of poles is at least two, for there are at least two points at which the level surfaces are tangents to the surface of the sphere; but there may be a far greater number. Suppose, for instance, that there are two poles P and P′ (Fig. 93) situate in the

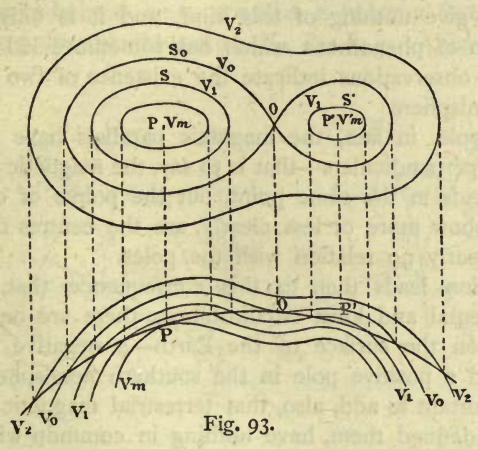

Fig. 93.

positive region—that is to say, on the southern hemisphere. These poles might belong to the same level surface which had two points of contact with the surface of the sphere; but more generally we shall consider them as belonging to two different surfaces of potentials V_m and V'_m, V_m being greater than V'_m.

Since the points P and P′ are points of maximum, the potential decreases in all directions around each of them, and we may always choose a value V_1 of potential lower than V'_m, such that the intersection of the surface V_1 with the sphere, gives two closed curves S and S′, insulated from each other, and each of which surrounds one of the points; we may then take a value V_2 so small that the same curve of intersection comprises the two points.

By making the potential vary continuously from V_1 to V_2, we shall find a value V_0 for which the two curves, previously separated, come in contact, and merge into a single one S_0; the junction may take place either by a single point of intersection, as in Fig. 93, or by a greater number of points of intersection or of contact. Let O be one of these points. It is first of all clear that the horizontal component there is null, and that therefore the point corresponds to the ordinary definition of poles; it is, however, to be observed that as we move in certain directions we get increasing, and in other directions decreasing, potentials; for the former directions the point O would act like a south pole, and for the second as a north pole. This is what we may call a *false pole.*

There cannot thus be two distinct poles in the same hemisphere without there being at the same time at least one false pole. But observations give nothing of this kind, and it is only an inexact interpretation of phenomena which has sometimes led to the conclusion, that observations indicate the existence of two poles in the northern hemisphere.

Near a pole, in fact, the magnetic parallels have an elliptical shape; their perpendiculars—that is to say, the magnetic meridians—do not coincide in the same point, but the points of convergence, which they show more or less clearly, are the centres of curvature, and have clearly no relation with the poles.

Observation leads then to this consequence, that, apart from purely accidental and local circumstances, there are only two magnetic poles on the surface of the Earth—a negative pole in the northern and a positive pole in the southern hemisphere.

It is important to add, also, that terrestrial magnetic poles, such as we have defined them, have nothing in common with magnetic poles, properly so called, considered as centres of gravity of positive and negative magnetic masses. The magnetic axis of the Earth is the right line along which the sum of the projections of the magnetic moments of the various elements is a maximum (297).

437. PROPERTIES OF A CLOSED POLYGON.—We know that if we move a magnetic mass equal to unity from a point P_1 where the potential is V_1, to a point P_2 where it is V_2, and if we denote by F the force, by ds the element of the path described by the mass, and by ϵ the angle of the force with the element, the magnetic work is expressed by the equation

$$V_1 - V_2 = \int_{P_1}^{P_2} F ds \cos \epsilon.$$

This work is independent of the path traversed, and is zero whenever we return to the original level surface by making the mass describe any given closed curve. Suppose that the two points P_1 and P_2 are situate on the surface of the Earth, and that the mass is displaced along this surface; the work of the vertical component is zero at each instant, the expression for the work only depends on the horizontal component H, and reduces to

$$(2) \qquad V_1 - V_2 = \int_{P_1}^{P_2} H ds \cos \epsilon,$$

the integral of the second member being zero whenever the mass is made to describe a closed circuit.

That being admitted, let us consider a polygon formed of great circles passing through the points P_0, P_1, P_2.... (Fig. 94). Trace

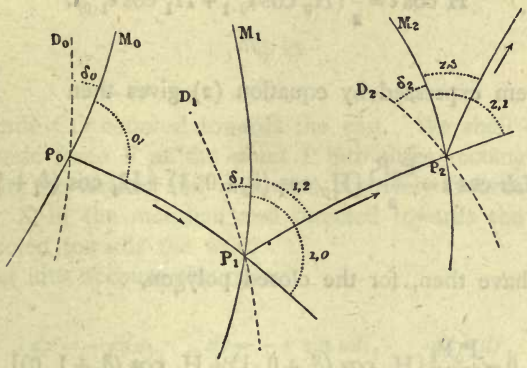

Fig. 94.

at these various points the geographical meridians P_0M_0, P_1M_1, P_2M_2,.... and the magnetic meridians P_0D_0, P_1D_1, P_2D_2...

Let

δ_0, δ_1, δ_2... be the declinations reckoned positively from north to west;

0.1 the azimuth of the arc P_0P_1 at the point P_0, this azimuth being counted positively from north to east;

1.0 the azimuth of the arc P_0P_1 at the point P_1 counted positively in the same direction;

$\epsilon_{0.1}$, $\epsilon_{1.0}$ the values of the angles ϵ at these various points.

We have

$$\begin{aligned}
&\text{At the point } P_0, \quad \epsilon_{0.1} = \delta_0 + 0.1\,; \\
&\text{,, ,, ,, } P_1, \quad \epsilon_{1.0} = \delta_1 + 1.0, \\
&\qquad\qquad\quad \epsilon_{1.2} = \delta_1 + 1.2\,; \\
&\text{,, ,, ,, } P_2, \quad \epsilon_{2.1} = \delta_2 + 2.1, \\
&\qquad\qquad\quad \epsilon_{2.3} = \delta_2 + 2.3\,; \text{ etc.}
\end{aligned}$$

On the side P_0P_1 the horizontal component H is not constant either in magnitude or in direction; nevertheless, if this side is very small compared with the dimensions of the terrestrial globe, we may assume that the value of H is constant, equal to the mean of the values which it has at the points P_0 and P_1, and put

$$H \cos \epsilon = \frac{1}{2}(H_0 \cos \epsilon_{0.1} + H_1 \cos \epsilon_{1.0}).$$

The theorem expressed by equation (2) gives then

$$\int_{P_0}^{P_1} H ds \cos \epsilon = \frac{P_0P_1}{2}[H_0 \cos (\delta_0 + 0.1) + H_1 \cos (\delta_1 + 1.0)].$$

We shall have then, for the closed polygon,

$$\begin{aligned}
0 = &\frac{P_0P_1}{2}[H_0 \cos (\delta_0 + 0.1) + H_1 \cos (\delta_1 + 1.0)] \\
&+ \frac{P_1P_2}{2}[H_1 \cos (\delta_1 + 1.2) + H_2 \cos (\delta_2 + 2.1)] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \frac{P_nP_0}{2}[H_n \cos (\delta_n + n.0) + H_0 \cos (\delta_0 + 0.n)].
\end{aligned} \tag{3}$$

Applying this equation to the triangle formed by the stations at Paris, Göttingen, and Milan, and taking as unknown the value H_0 at Paris, Gauss found by calculation the value $H_0 = 0.517$, while observation gave 0.518.

438. INTRODUCTION OF GEOGRAPHICAL CO-ORDINATES.—Let us consider any given point P at a distance r from the centre of the Earth; let u (Fig. 95) be the complement P'D of the latitude and l

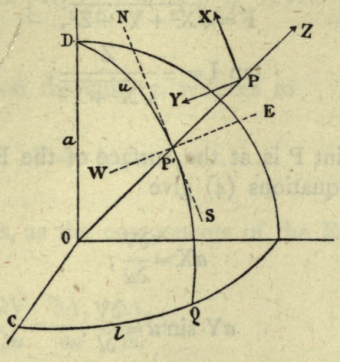

Fig. 95.

the longitude CQ counted towards the east. We shall decompose the magnetic force F at the point P into three rectangular forces, one Z along the vertical and measured positively towards the zenith, the other X in the meridian and directed towards the north, the third directed towards the west.

Taking into account the ratios

$$dx = -rdu, \qquad dy = -r \sin u dl, \qquad dz = dr,$$

the components of the force become

$$\text{(4)} \qquad \begin{aligned} X &= -\frac{\partial V}{\partial x} = -\frac{1}{r}\frac{\partial V}{\partial u}, \\ Y &= -\frac{\partial V}{\partial y} = \frac{1}{r \sin u}\frac{\partial V}{\partial l}, \\ Z &= -\frac{\partial V}{\partial z} = -\frac{\partial V}{\partial r}. \end{aligned}$$

We have, moreover,

$$H=\sqrt{X^2+Y^2},$$

$$\tan\delta=\frac{Y}{X},$$

$$F=\sqrt{X^2+Y^2+Z^2},$$

$$\tan I=-\frac{Z}{\sqrt{X^2+Y^2}}.$$

When the point P is at the surface of the Earth at P', we must take $r=a$, and equations (4) give

$$aX=\frac{\partial V}{\partial u},$$

$$aY\sin u=\frac{\partial V}{\partial l}. \tag{5}$$

Since, further, we have

$$\frac{\partial^2 V}{\partial u\,\partial l}=\frac{\partial^2 V}{\partial l\,\partial u}$$

we get

$$\frac{\partial X}{\partial l}=\frac{\partial\,(Y\sin u)}{\partial u},$$

and, therefore,

$$Y\sin u=\int_0^u\frac{\partial X}{\partial l}\,du+f(l).$$

For $u=0$—that is to say, at the north pole—we have $Y\sin u=0$, and therefore $f(l)=0$. We get then, finally,

$$Y\sin u=\int_0^u\frac{\partial X}{\partial l}\,du. \tag{6}$$

We are thus led to the remarkable theorem of Gauss:

If for all points of the surface of the earth we know the horizontal component directed towards the north, that is sufficient to give us the horizontal component turned towards the west, and therefore the total horizontal component.

439. EXPRESSION OF POTENTIAL.—Whatever may be the magnetisation of the Earth, the external potential may be represented, as we have seen (369), by the expression

$$V = a\left[A_1\left(\frac{a}{r}\right)^2 + A_2\left(\frac{a}{r}\right)^3 + \ldots\ldots\right],$$

which, for a point on the surface, reduces to

$$V = a\,(A_1 + A_2 + \ldots\ldots).$$

We deduce from this, as the components of the Earth's magnetism,

$$X = -\frac{1}{a}\frac{\partial V}{\partial u} = \frac{\partial A_1}{\partial u} + \frac{\partial A_2}{\partial u} + \ldots\ldots,$$

$$(7) \qquad Y = \frac{1}{a\sin u}\frac{\partial V}{\partial l} = \frac{1}{\sin u}\left[\frac{\partial A_1}{\partial l} + \frac{\partial A_2}{\partial l} + \ldots\ldots\right],$$

$$Z = -\frac{\partial V}{\partial r} = 2A_1 + 3A_2 + 4A_3 + \ldots\ldots$$

The coefficients A_1, A_2, A_3, are functions of the two angles l and u. A_n is expressed (368) by $2n+1$ terms in sines and cosines. Hence, if we wish to represent the condition of the Earth by a series of this form, we must determine three numerical coefficients for A_1, five for A_2, seven for A_3, etc.

Gauss found that, in the then existing condition of magnetic determinations, it was useless to push the development beyond the fourth term, so that there are still twenty-four numerical coefficients to calculate.

Every point of the surface gives three equations by the values of the components X, Y, Z; hence, if we know these three elements at any eight places in the earth, we have a complete solution of the problem. In order to avoid errors arising from neglected terms, and from inexact observations, Gauss applied the method of least squares to the data for eighty-four points, taken on twelve equidistant meridians, and seven parallels. The results thus obtained were then applied to ninety-nine other points.

The formulas calculated by Gauss assign to the two poles the following positions for the year 1838:—

North Pole	latitude	73° 35′	longitude	95° 39′ W,
South Pole	„	72° 35′	„	152° 30′ E;

they are, as will be seen, far from corresponding to the ends of the same diameter.

The true magnetic axis, determined by the condition that the sum of the projections of the moments is a maximum, is parallel to the terrestrial diameter which corresponds to that point in the northern hemisphere the latitude of which is 77° 50′, and the longitude 63° 31′ W. Its direction does not coincide exactly with the line of the poles.

This direction is that for which the coefficient A_1 has its maximum value (370). The magnetic moment of the Earth is equal to a^3K. Comparing this moment with that of a magnetised steel bar, which weighed about 500 grammes, and had been used in the absolute determination of the Earth's magnetism, Gauss found that it was about 8.10^{21} times as great. If we suppose the Earth to be uniformly magnetised, it follows from this number that the magnetic moment of each cube metre of the terrestrial globe is the same as that of eight of the magnets used by Gauss. Assuming that the magnetisation of the bar was also uniform, the intensity of its magnetisation would be about 2200 times that of the terrestrial globe.

440. IS THE MAGNETISM OF THE EARTH IN THE INTERIOR ONLY?—It may be observed that if the acting masses were in part in the interior and part outside, the potential might be expressed by the sum of two series

$$V = A_1\left(\frac{a}{r}\right)^2 + A_2\left(\frac{a}{r}\right)^3 + \ldots\ldots + A_n\left(\frac{a}{r}\right)^{n+1} \ldots\ldots$$

$$+ B_1\left(\frac{r}{a}\right) + B_2\left(\frac{r}{a}\right)^2 + \ldots\ldots + B_n\left(\frac{r}{a}\right)^n \ldots\ldots,$$

the former relative to the internal masses, and the second to the external masses. Denoting by V_n the general term of the development, we should have then

$$V_n = A_n\left(\frac{a}{r}\right)^{n+1} + B_n\left(\frac{r}{a}\right)^n;$$

from which is deduced

$$\frac{dV_n}{dr} = -\frac{n+1}{r} A_n \left(\frac{a}{r}\right)^{n+1} + \frac{n}{a} B_n \left(\frac{r}{a}\right)^{n-1}.$$

For a point on the surface we have simply

$$V_n = A_n + B_n,$$

$$\frac{dV}{dr} = -\frac{n+1}{a} A_n + \frac{n}{a} B_n. \tag{8}$$

The vertical component

$$Z = -\frac{dV}{dr} = -\left(\frac{dV_1}{dr} + \frac{dV_2}{dr} + \ldots\right) = -(Z_1 + Z_2 \ldots + Z_n \ldots)$$

has for its general term

$$Z_n = \frac{1}{a}[nB_n - (n+1) A_n]. \tag{9}$$

This equation, combined with the preceding one (8), gives

$$B_n = \frac{1}{2n+1}[aZ_n + (n+1) V_n], \qquad A_n = \frac{1}{2n+1}[nV_n - aZ_n];$$

and we may thus easily separate the effect due to internal masses from those produced by external masses.

The calculations of Gauss having shewn that the observations are satisfied by means of the single coefficients A, it follows that the coefficients B are virtually null; hence no sensible part of the terrestrial action is due to external magnetic masses.

441. Influence of the Sun and Moon.—There are, however, certain periodical variations in the elements of terrestrial magnetism, which appear to be connected with the apparent motions of the Sun and Moon, or at any rate to depend on certain accessory phenomena such as the spots of the Sun. The influence of these bodies can scarcely be doubted; everything leads, however, to the belief that they do not act directly as magnetic bodies, but that their influence is indirect, and only modifies the magnetic condition of the terrestrial globe.

A star, whatever may be the distribution of its magnetism, is, in fact, equivalent for very distant points to an infinitely small magnet, or to a sphere magnetised uniformly.

Let us denote by:

I the mean intensity of the Earth's magnetisation;
R its radius;
I′ the mean intensity of the magnetisation of a star;
R′ its radius;
ϖ' its magnetic moment;
D its distance from the Earth.

The value of the action of the Earth at the equator, where it is a minimum, is (153)

$$T_e = \frac{\varpi}{R^3} = \frac{4}{3}\pi I.$$

If we suppose that the line of the poles of the star in question is directed towards the Earth, which is the most favourable case, the force F_p, which it will exert on the Earth, will be (153)

$$F_p = 2\frac{\varpi'}{D^3} = 2 \cdot \frac{4}{3}\pi I' \left(\frac{R'}{D}\right)^3.$$

The ratio of the polar action of the star in question, to the equatorial action of the Earth, is

$$\frac{F_p}{T_e} = 2\frac{I'}{I}\left(\frac{R'}{D}\right)^3 = \frac{1}{4}\frac{I'}{I}\left(\frac{2R'}{D}\right)^3.$$

The ratio is therefore proportional to the magnetisation of the star and to the cube of its apparent diameter.

The apparant diameter of the Sun, and that of the Moon, are about 30′—that is to say, less than 0.01—so that we have

$$\frac{F_p}{T_e} < \frac{1}{4}\frac{I'}{I} 10^{-6}.$$

If these stars are magnetised like the Earth, the greatest variation which they could produce at the equator, on the declination, is less therefore than $\frac{10^{-6}}{4}$ or $\frac{1''}{20}$, that is to say, absolutely inappreciable. To have variations of 10′, such as are frequently met with, the intensity of the magnetisation of the Sun and of the Moon, must be 12,000 times as great as that of the Earth. Now the most powerfully magnetised steel has not 10,000 times the intensity of the Earth; hence, to produce a deviation of 10′, the Sun and Moon should be more powerfully magnetised than the best steel magnets.

The same conclusions result from supposing that the Moon, for instance, is magnetised by the Earth. If the Moon is at the equator, the action which it experiences from the Earth is

$$f=\frac{4}{3}\pi \mathrm{I}\left(\frac{\mathrm{R}}{\mathrm{D}}\right)^3,$$

and the value of the intensity of the magnetisation is

$$\mathrm{I}'=kf=\frac{4}{3}\pi \mathrm{I}k\left(\frac{\mathrm{R}}{\mathrm{D}}\right)^3.$$

From this is deduced

$$\frac{\mathrm{I}'}{\mathrm{I}}=k\frac{4}{3}\pi\left(\frac{\mathrm{R}}{\mathrm{D}}\right)^3=k\frac{4}{3}\pi\left(\frac{1}{60}\right)^3=\frac{k}{54{,}000}.$$

Whatever value we assume for the coefficient k—if even we compare the Moon with the very softest iron—the ratio of the magnetisations will be always very small, and the reaction of the Moon upon the Earth may be completely neglected. Still more must this be the case with the Sun.

PART IV.—ELECTROMAGNETISM.

CHAPTER I.

CURRENTS AND MAGNETIC SHELLS.

442. Œrsted's Experiment.—Older experiments on electrical discharges had already shewn that the passage of a current in a conducting wire could modify the magnetism of a steel needle. These phenomena, to which only small importance was attached, were a first indication of the relations which existed between electricity and magnetism. It is only since 1820, in consequence of Œrsted's experiment, that the existence of these relations has been made completely evident by the immortal researches of Ampère.

When a straight conductor traversed by a current is brought near a magnetised needle, the needle is, in general, deflected from its position. In order to explain in all cases the somewhat complicated effects which are observed according to the relative positions of the magnet and the current, Ampère gave a very simple rule: Suppose an observer placed in the wire in such a manner that the current enters at his feet and emerges at his head; the observer, turning his face to the needle, always sees the North pole turn towards his left, which for the future we shall call the *left of the current.* If the needle were freed from the action of the Earth, and of any other action than that of the current, it would set at right angles with the current.

443. Magnetic Field of a Current.—The fundamental fact which results from Œrsted's experiment, is that an electrical current of any given form creates about itself a true *magnetic field.*

This field possesses all the properties observed in an ordinary magnetic field, for the actions which it exerts at any point on equal magnetic masses of opposite signs are equal and directly

opposite. The force, moreover, is proportional to the magnetic mass in question, for if we put near the current a small needle, which at the same time is under the action of the Earth, the direction which it takes is independent of its magnetic moment; the resultant of the two forces which arise from the terrestrial field, and from the field created by the current, has thus itself a fixed direction, and therefore the two forces maintain a constant ratio.

The action of the current also changes its sign, without changing its magnitude, when the direction of the current is simply reversed; thus, when the conducting wire is bent upon itself, the two portions close to each other, which are traversed by equal currents in opposite directions, have no action on the pole of a magnetised needle.

The existence of the field produced by the current may be rendered evident by the ordinary method of magnetic images. Thus, if iron filings are scattered on a sheet of paper traversed at right angles in its centre by a rectilinear current, the filings are seen to arrange themselves in concentric circles about the path of the current. We conclude from this that the lines of force are circumferences whose centre is the axis of the current. The force, therefore, is perpendicular at each point to the plane passing through this point and the current; it is, moreover, turned to the left of the observer in Ampère's rule.

The successive equipotential surfaces obtained about a rectilinear current are thus formed by a series of planes passing through the axis of the wire, and making equal angles with each other. The same is the case near any given current, so that the equipotential surfaces are formed about each portion of the wire, making equal angles with each other.

444. Action of a Rectilinear Current on a Pole.—Experiments of Biot and Savart.—Biot and Savart determined experimentally the magnitude of the force at each point. They examined the action of a vertical current on a small horizontal magnetic needle placed at various distances on a right line passing through the current, and perpendicular to the magnetic meridian. In these conditions the resultant force is the sum of the horizontal component H of the Earth's field, and of the force ϕ of the current.

The needle is first caused to oscillate under the influence of the Earth alone, then at distances a and a' from the wire under the combined influence of the Earth and of the current. If n, N, and

N′ are the numbers of oscillations of the needle in a given time t in the three experiments, then if K is a constant depending on the magnetisation of the needle and on its moment of inertia, we have

$$n^2 = KH,$$
$$N^2 = K(H+\phi),$$
$$N'^2 = K(H+\phi').$$

From which is deduced

$$\frac{\phi}{\phi'} = \frac{N^2 - n^2}{N'^2 - n^2}.$$

But experiment showed that by employing the method of alternate distances to eliminate the effect of variations in the strength of the current, the following ratio was always obtained:

$$\frac{N^2 - n^2}{N'^2 - n^2} = \frac{a'}{a}.$$

It follows from this that $\phi a = \phi' a'$, that is to say, that the action of the current on a point is inversely as the distance.

On the other hand, experiments made on the discharge of batteries—those of Colladon and of Faraday particularly—and the more accurate measurements made with the voltameter, have shown that the magnetic action of a current is proportional to the quantity of electricity which flows during unit time—that is to say, to the intensity i of the current.

The action exerted by a rectilinear current on a magnetic mass m at a distance a, may then be represented by the expression

$$\phi = \frac{kim}{a}, \tag{1}$$

in which k is a coefficient to be determined.

The action observed in this experiment, as well as in that of Œrsted, is always the action of a closed current; but it is easy to see that if the rectilinear portion is sufficiently great, and the rest of the current sufficiently distant, the action of this latter part is inappreciable, and the effect observed only depends on the nearest part. The action of the rectilinear portion may then be considered as

equal to that of an unlimited rectilinear current. Hence we arrive at the following law of Biot and Savart:

The action of an unlimited rectilinear current on a pole is perpendicular to the plane passing through the current and the pole, is directed towards the left of the current, and is inversely as the distance of the current from the pole.

A simpler experiment, at any rate in theory, leads to the same result. Suppose that a portion of the circuit is vertical, and a magnet placed in any given way, upon an apparatus movable about an axis which coincides with that of the current. It will be seen that the movable system is at rest for all positions of the magnet, whatever be the direction and strength of the current. It follows hence that the sum of the moments, in reference to the axis, of the actions exerted on the different masses of the magnet, is null.

If m is the magnetic mass at a distance a from the axis, we shall have

$$\sum m\phi a = 0.$$

If we suppose the magnet reduced to two masses $\pm m$ equal and of contrary signs, at the distances a and a' from the current, the equation reduces to

$$m(\phi a - \phi' a') = 0, \qquad \text{or} \qquad \phi a = \text{const.},$$

that is to say, to Biot and Savart's law.

The experiment carries with it its own verification, for if we cease to make the axis of rotation coincide with the axis of the current, the system is displaced, and tends to turn in one or the other direction to obtain its position of equilibrium.

445. Potential of an Unlimited Rectilinear Current.—We shall proceed to show that the magnetic field of a current is defined by a potential—that is to say, by a function whose partial differentials, in reference to the axis of the co-ordinates, represent the respective components of the force taken with contrary signs.

In the case of a rectilinear current, the equipotential surfaces are planes passing through the current. Let us take the current for the z-axis, and a plane perpendicular to the current passing through the point P (Fig. 96) for the plane xy. If we suppose that the current goes behind the figure, the force ϕ at a point P of the plane, from Ampère's rule, is perpendicular to PO, and would tend to turn this

point about the current in the direction of the hands of a watch. Let α denote the angle POy. For a very small displacement PP′ in the direction of the force, the work on a positive mass equal to unity will be

$$dT = \phi \times PP' = \phi a d\alpha = ki d\alpha.$$

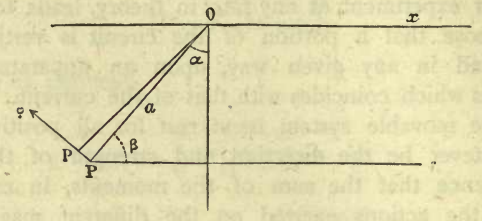

Fig. 96.

As the angle β, which the right line PO makes with a parallel Px' to the axis, is complementary to α, we may write

$$dT = -ki d\beta.$$

This work is equal to the corresponding fall of potential $-dV$; we deduce therefrom

$$dV = ki d\beta,$$

and therefore

$$V = ki\beta + \text{const.}$$

We may observe that the angle β is the rectilinear angle of the dihedral angle of two planes drawn from the point P, one in the current, the other parallel to the current and to the x-axis; twice this angle β measures the surface ω of the spindle, which the dihedron cuts through a sphere of unit radius with its centre at P; we may accordingly write

$$V = \frac{ki}{2}\omega + \text{const.}$$

The surface ω is evidently the solid angle under which the plane of xz is seen from the point P, unlimited in one direction, and bounded in the other by the current—that is to say, the apparent surface of this plane.

We conclude from this that *the potential of an unlimited rectilinear current at a point is, within a constant, proportional to the product of the strength by the apparent surface of a plane, unlimited in one direction, and bounded in the other by the current.*

In order to determine the sign of this apparent surface, we must remember that, in practice, the unlimited rectilinear current necessarily forms part of a closed circuit, and that if the non-rectilinear portion is very distant from the point P, the angle under which the whole circuit is seen, which we may suppose plane, only differs by an inappreciable quantity from the unlimited plane of which it forms part. We shall call that face of the current, which is on the left of the observer placed in the current, and who is looking inwards, the *positive face;* the *negative face* being that on his right; and we shall take the angle ω positive or negative, according as the positive or the negative face is seen from the point P.

446. The Potential of an Unlimited Current is not a Simple Function of the Co-ordinates.—At a given point, the angle ω only gives the value of the potential of an unlimited current to within a constant. It is easy to see what is the significance of this constant. Suppose that a unit positive mass taken at the point P (Fig. 96) describes a circumference about the point O, in the direction of the force, and reverts to its original position. The angle ω has resumed the same value, but the force ϕ has performed a work $\phi 2\pi a$—that is to say, $2\pi ki$ or $4\pi \frac{ki}{2}$, and this mass has traversed the plane of the current by the negative face. For n turns of the mass, the work would be equal to $4\pi n \frac{ki}{2}$, and the potential would have varied by the same quantity $-4\pi n \frac{ki}{2}$.

On the other hand, the expression $\frac{ki}{2}\omega$ is the work which must be expended against magnetic forces, in order to bring this mass from infinity to the point P, without traversing the plane of the current.

If then, by analogy with the properties of magnetic shells, we call the potential at a point, the work necessary to bring a positive magnetic mass equal to unity from an infinite distance, this potential is expressed by

$$(2) \qquad V = \frac{ki}{2}\omega - 4\pi n \frac{ki}{2} = \frac{ki}{2}(\omega - 4\pi n).$$

The magnetic potential of the current at a point is not, therefore, a simple function of the co-ordinates, but a function having an infinity of values, which differ from each other by a multiple of $4\pi \frac{ki}{2}$; that is to say, of the work which would be represented by the complete rotation about the current of a magnetic mass equal to unity. This property may be easily generalised.

447. Potential of an Angular Current.—Let us consider two unlimited currents AA′ and BB′ (Fig. 97) of the same strength,

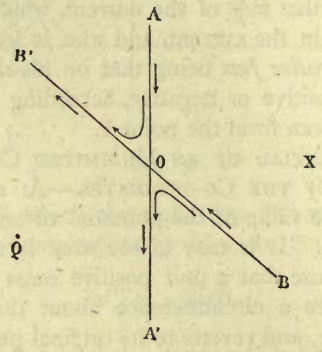

Fig. 97.

situated in the same plane, and moving in the directions indicated by the arrows. Let Q be the projection of the pole P on this plane. The potential at P of the current AA′ is proportional to the apparent surface of the plane AA′X; that of BB′ is proportional to the apparent surface of the plane BB′X.

With the actual direction of the current, and assuming that their planes extend indefinitely on the right, these two apparent surfaces must be taken with contrary signs, and the resultant potential is equal to their difference. But the part in common, AOBX, disappears; the potential is therefore proportional to the apparent surface of the angle BOA′, diminished by the apparent surface of the angle AOB′.

On the other hand, the system of two unlimited currents is identical with that of two angular currents BOA′ and AOB′, the first of which turns its positive face to the front, and the second its negative face.

We may accordingly assert that the potential at a point P of an angular current, such as BOA′, is proportional to its apparent surface,

within a function of the co-ordinates of the apex of the angle; a function whose sign depends on the sign of the surface turned towards the point, and which, moreover, would disappear in applications.

448. POTENTIAL OF A TRIANGULAR CURRENT.—Let us suppose further that in the same plane there is a third current CC′ (Fig. 98), identical with the former, and forming with it a triangle *abc*.

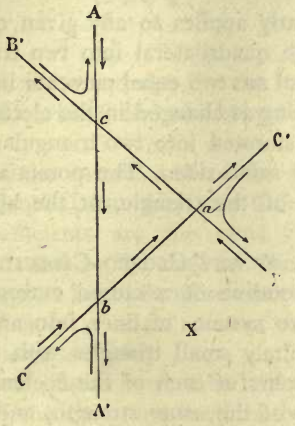

Fig. 98.

The potential at P of the two former is proportional to the apparent surface of the angle B*c*A′, less that of the angle A*c*B′. The potential of the current CC′ is proportional to the apparent surface of the plane CC′X taken with the − sign. If we add together the effects of the three currents, the part in common B*ab*A′ disappears, and finally there remain, in the expression of the potential, the apparent surface of the triangle *abc*, and that of the external angles A*c*B′, C*b*A′, and B*a*C′, these latter being all taken negatively.

Let us add to the system three angular currents of the same strength represented by bent arrows; they will introduce into the potential the apparent surfaces of these same angles taken positively this time, so that only the apparent surface of the triangle will remain. Of all the currents only that circulating round the angle will remain, for each of the external lines is traversed by equa currents of opposite signs.

Thus *the potential at a point* P *of a closed triangular current is proportional, within a constant, to the apparent surface of the triangle which the current encloses, or to the solid angle under which the triangle is seen from the point* P.

If this angle be called ω, we have

$$V = \frac{ki}{2}\omega + \text{const.}$$

The theorem clearly applies to any given quadrilateral; for we may always divide the quadrilateral into two triangles, and suppose that along the diagonal are two equal currents in opposite directions. By this addition nothing is changed in the electrical system, and the given current is transformed into two triangular currents with their positive faces on the same side. The potential is the sum of the two apparent angles of the triangle, or the apparent angle of the quadrilateral.

449. Potential of any Closed Circuit.—We can draw any surface through the outline of a closed current, and suppose this surface divided by two systems of lines, into any number of quadrilaterals, and of infinitely small triangles with rectilinear sides. If we suppose the contours of each of these elementary figures to be traversed by currents of the same strength, and the same direction as the principal current, we should obtain a system of closed currents which will be equivalent to the given current, since each of the interior lines is traversed by two equal currents in contrary directions, and the only effective portions are those, the whole system of which forms the given current. As all the elementary currents have their faces turned in the same direction, the potential of the system is proportional to the sum of the apparent surfaces of the elementary currents—that is to say, to the apparent surface of the proposed current.

Hence *the potential at any point* P *of any closed current is given, to within a constant, by the solid angle under which the contour of the current is seen from the point* P.

450. Equivalence of a Closed Current and of a Magnetic Shell.—Ampère's Theorem.—Let ω be the value of the solid angle under which the contour of the current is seen from the point P, then from the preceding theorem

$$V = \frac{ki}{2}\omega + \text{const.} \tag{3}$$

For a magnetic shell of power Φ, which would be bounded by the same outline (329), we should have

$$V = \Phi\omega.$$

The two potentials will then be equal, within a constant, provided that

$$\frac{ki}{2} = \Phi = I, \tag{4}$$

the symbol I being a new expression for the strength of the current defined by this condition itself, and which we call the *electromagnetic strength.*

The potentials of the current and of the shell for which $I = \Phi$ are not absolutely identical, but they only differ by a constant, and their differential coefficients are the same. Hence the actions exerted by the current and by the shell are the same for each point of the field. We are thus led to Ampère's celebrated theorem:

The magnetic action of a closed current is equal to that of a magnetic shell of the same contour.

The positive forces of the current and of the shell correspond, and are on the left of the observer placed in the current, and who is looking towards the interior of the circuit.

We have deduced this important theorem from Biot and Savart's experiment, but we might consider it as an experimental fact verified in all its consequences, and accept it as a starting point to deduce from it all the magnetic properties of currents.

451. Remarks on the Equivalence of a Closed Current and a Magnetic Shell.—It is important to insist on the conditions of the equivalence of the current and of the shell. We have seen that with a shell the force is not a continuous function of the co-ordinates; it is constant in the interior of the shell, and changes its sign when one of the surfaces is passed through; the lines of force start on each side of the positive face, and are absorbed by the negative. These sudden changes do not take place in the case of a closed current; the force is a continuous function of co-ordinates, and the lines of force are closed curves which do not touch the circuit, and do not encounter any acting mass. It will be seen that this may be the case, without any contradiction; for the shell equivalent to the current is only under the condition of being bounded by the same contour, and we may suppose that when a

magnetic mass is displaced in the vicinity of a current, the equivalent shell is being constantly deformed, and recedes before it without ever being met.

The analogy of the two systems becomes closer if instead of considering the magnetic force of a shell we consider the induction. We know, in fact, that magnetic induction is a continuous function of co-ordinates, that the flow of induction is maintained throughout the entire extent of an orthogonal tube, and that the force and magnetic induction have the same value for any point outside the magnetised media. As a particular case, the magnetic induction in the thickness of a shell is identical with the force which would be produced there, if the shell, while still retaining the same contour, and the same magnetic power, were deformed in such a manner as no longer to include the point in question, and this force is equal to that of an equivalent current which went along the contour. This is evidently the same also for any system of currents; from which is deduced the general law:

Any system of closed currents is equivalent to a magnetic system, and the action of currents at a point is identical with the induction, at the same point, of the equivalent magnetic system.

452. Relative Energy of a Magnetic System and a Current.—The potential of a current at a point P, is within a constant equal to $-I\omega$, if we denote by ω the solid angle under which the negative face of the current is seen. The product $-mI\omega$ is the work which would be expended in bringing a magnetic mass equal to m, from an infinite distance to this point, without traversing a continuous surface bounded by the current. The potential energy of the mass m at the point P is then, within a constant, equal to $-mI\omega$.

If this mass has passed the surface of the current n times by the positive face to arrive at the point P, the work $mI4\pi$ must each time have been expended; the total work is then

$$mI(4\pi n - \omega).$$

Conversely, if the mass is left to itself, it tends to turn indefinitely around the current, and at each turn expends an amount of energy equal to $m4\pi I$.

This continuity of motion is not possible with two magnetic systems, for the potential is then a determinate function of the co-ordinates; it would, moreover, be inconsistent with the principle of the conservation of energy. With currents the movement may

be continuous, for an extraneous energy necessarily comes into play in the phenomena, such as that of the chemical actions which take place in batteries.

If Q be the flow of force of the magnetic system, which traverses the surface of the current, entering by the negative face, the relative energy of the two systems is expressed, to within a constant, by

$$W = -IQ. \tag{5}$$

When the magnetic system is left to itself, the work $d'T$ of the magnetic forces, for any infinitely small displacement, is equal and of opposite sign to the change in the energy, and we have

$$d'T + dW = 0,$$

or

$$d'T = I d'Q.$$

The motion of the system takes place in such a way that the value of Q tends towards a maximum.

For two successive positions characterized by the indices 1 and 2, the work, to within a constant, will be

$$T_1^2 = I(Q_2 - Q_1).$$

It is important to remark that in a general way the difference $Q_2 - Q_1$ depends not only on the final and initial positions of the magnetic system, but also on the path pursued by each mass; for the product $m4\pi I$ should be added to the work of all those which should have surrounded one of the branches of the current. If a long and flexible uniform magnet, for instance, were placed near a current, the positive pole would turn in one direction indefinitely about the current, and the negative pole in the contrary direction.

Nevertheless, if the contour of the current is rigid, as well as the magnetic system, all the masses which constitute the magnet will necessarily make the same number of turns, and in the same time, and the corresponding work is equal to $n4\pi I\sum m$. As the total mass of a magnet is always zero, the work of any given displacement only depends on the initial and final position of the system, and not of the path traversed; in this case the work is null when the magnet reverts to its original position.

It is therefore impossible to obtain the continuous motion of a magnet by a current which traverses a rigid system; the reciprocal

action of the two systems is then identical with that of two magnets. The maximum and minimum values of the flow of force Q correspond to positions of relative equilibrium, stable in the former case and unstable in the latter. The motion may be continuous, on the contrary, if the circuit can be deformed; if it contains, for instance, liquid portions, sliding contacts, or if it can be broken in certain parts while the magnet is being displaced.

453. Reciprocal Action of Two Closed Currents.—It may be asked whether a closed current and a shell, which are equivalent with respect to any magnetic system, are so towards another current? Thus the current C_1 and the shell S_1 of the same contour, are equivalent in their action upon the magnetic system M_2; suppose that this magnetic system is a shell S_2; the reciprocal action which is exerted between S_1 and S_2 is identical with that which is exerted between S_1 and the current C_2, which is equivalent to S_2; but is this latter action the same as that which would be exerted between the two currents C_1 and C_2? The affirmative seems probable; but this is only an induction, and it would be easy to find examples, for which the same mode of reasoning would lead to conclusions which are manifestly erroneous. Thus, under conditions suitably chosen, it might happen that the actions exerted upon a magnet by a magnet and by a piece of soft iron are the same; we could not conclude from this that the soft iron and the magnet would be equivalent towards another piece of soft iron.

It is therefore as an experimental result, and not as a necessary deduction from theory, that we shall assume the following theorem of Ampère:

The reciprocal action of two closed currents is identical with that of two magnetic shells respectively equivalent to each of them.

454. Relative Energy of Two Currents.—The value of the potential energy of two magnetic shells (341) is

$$W = -\Phi\Phi' M.$$

From Ampère's theorem, that of two closed currents will be expressed, to within a constant, by the formula

$$(6) \qquad W = -II'M,$$

in which I and I′ are the strength of the two currents, and M the flow of force which, starting from one of the circuits, traverses the other by its negative face, the strength in each of them being equal to unity.

The work $d\mathrm{T}$ of the magnetic force corresponding to an infinitely small displacement will be given by the equation

$$d\mathrm{T} = -d\mathrm{W} = -\mathrm{H}'d\mathrm{M}. \tag{7}$$

455. ELECTROMAGNETIC ROTATION.—We have seen that the reciprocal action of a magnet and of a rigid current cannot produce a continuous motion. This would also be the case with two rigid currents, but the impossibility ceases if one of the systems can be deformed, and the preceding considerations enable us to give a simple explanation of most of the experiments of this kind.

Consider, for instance, an unlimited rectilinear current, the outline of which is at O (Fig. 99), and a magnet PP', one of whose poles P' can slide along a groove AB perpendicular to the current, while the other pole P may describe a circumference about the current, by means of a movable contact which opens the passage to it at each turn. The pole P will rotate for an unlimited time about

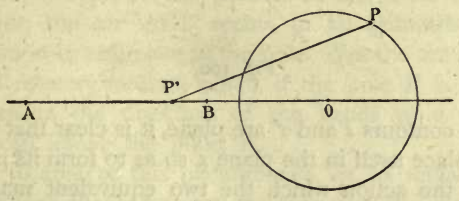

Fig. 99.

the current, and apart from friction the velocity will go on increasing, since the magnetic action of the current performs, at each turn, work equal to the product of $4\pi\mathrm{I}$ by the mass of the pole. A regular condition is established from the time in which the passive resistances balance the motive force. We shall see several examples of the same kind in one of the following chapters.

The action of a current on itself may also give rise to deformations, or to continuous motions.

Let ACB (Fig. 100) be a portion of a circuit movable about an axis passing through the two points A and B by which it is attached to the general circuit. We may suppose that the line A B is traversed by two currents in contrary directions, of the same intensity as the current itself, and may thus decompose the system into two distinct closed circuits s and s'.

If there were no other forces in the field than those arising from these two circuits, the relative energy of the two currents is

$$W = -I^2M.$$

As the energy tends towards a minimum, the movable part would displace itself, so that the flow of force M is a maximum.

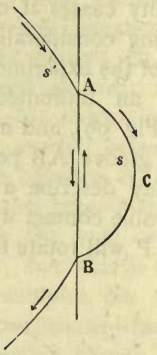

Fig. 100.

If the two contours s and s' are plane, it is clear that the movable part s would place itself in the plane s' so as to form its prolongation: it is seen by the action which the two equivalent magnetic shells would exert on each other.

If the general circuit consists of a flexible wire of a given length, the action of a current on itself would tend to give it the greatest surface—that is to say, to make it take the form of a circumference of a circle.

If the wire is elastic it will elongate until the elasticity balances the electromagnetic forces.

456. FARADAY'S EXPERIMENTS.—In certain cases the fundamental formula

$$W = -IQ$$

appears not to hold, as continuous movements can be produced, even when the flow of force through the system is zero or is invariable.

Let us consider, for instance, an arc ACB of a plane curve (Fig. 101) movable about a straight line AB, passing through the axis of a magnet PP', one of the ends being placed between the two

poles and the other without, and let us suppose that a current goes from the point A, to the point B, by the arc ACB. The flow of magnetic force from the pole P, which traverses the portion ACB of

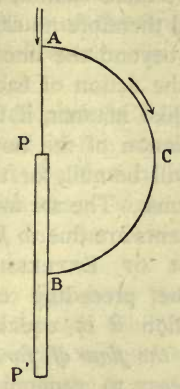

Fig. 101.

the circuit, seems null, for the pole is in the plane of the circuit; and, moreover, the arc ACB seems in all azimuths to have an identical position in reference to the pole. Yet the arc ACB acquires a continuous rotatory motion, which, if the pole P is a North pole, makes it turn in the direction of the hands of a watch for an observer placed above the point A.

In order to analyze this phenomenon let us replace the current by the equivalent shell; we may imagine that the movable part of this shell is made up of an infinitely elastic plate, which forms a concave surface behind the pole, and presents its negative face to the pole. As this surface tends to comprise a great part of the flow, it will move in the direction indicated, and the motion will be continuous, as the elastic sheet can fold upon itself indefinitely.

The change in the flow of the force for a rotation θ of the plane of the current, will be equal to $2m\theta$; the work of the electromagnetic forces relatively to this pole will be $2m\theta\mathrm{I}$ for the displacement θ, and for a complete turn, $4\pi m\mathrm{I}$.

The moment of the couple of rotation in reference to the axis is then expressed by

$$\frac{4\pi m\mathrm{I}}{2\pi} = 2m\mathrm{I};$$

it is to be observed that this moment is independent of the magnitude and shape of the arc.

The action of the lower pole P′ is obviously null; for any portion of the flow of force starting from this point, and which meets the sheet, necessarily cuts it twice, entering first by the positive, and then by the negative face; there cannot, therefore, be any variation of energy on this side, and therefore no cause of motion.

If the two poles were beyond the line AB, which joins the ends of the movable current, the action of each will be null, and there will be no rotation. In like manner, if the two poles were in the interval AB, the total variation of the flow of force relative to any displacement of the arc will be null, for the two poles will produce equal and opposite variations. The arc must, therefore, be at rest.

These various experiments are due to Faraday.

457. ANOTHER FORM OF EXPRESSION FOR THE ELECTROMAGNETIC WORK.—In the preceding example the work $2m\mathrm{I}\theta$ corresponding to a rotation θ is equal to the product *of the strength of the current by the flow of force cut by the arc* ACB *in the displacement.* It is easy to generalise the expression for the work in this new form.

Let us consider, in fact, a fixed magnetic system, in the field of which a current experiences any given displacement or deformation. Let s and s' be the two successive positions of the current (Fig 102),

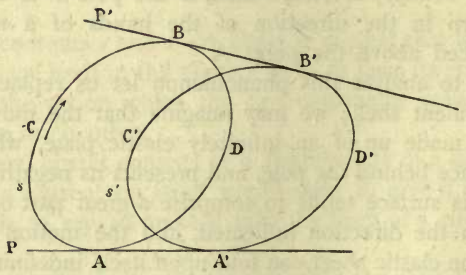

Fig. 102.

and Q and Q′ the flows of force which traverse the negative face in the two cases. The corresponding work of the electromagnetic forces is $\mathrm{I}(Q' - Q)$.

Draw two planes P and P′ tangential to the two positions of the circuit; join the points of contact AA′, and BB′, and denote by Q_1 and Q_2 the flow corresponding to the surfaces A′ACBB′C′ and A′ADBB′D′. We have evidently

$$Q' - Q = Q_2 - Q_1.$$

But Q_2 is the flow of force cut by the arc BDA, Q_1 the flow cut by the arc ACB in the displacement; hence we may say that the work of electromagnetic forces is equal to the excess of the flow cut by one of the portions of the circuit over the flow cut by the other. If the forces traverse the plane of the figure from front to back, the values of the flow are positive for the direction of the current indicated by the arrow.

For the elements of the curve ACB, the motion is to the right of an observer who is placed in the current, and who looks in the direction of the force, and the flow of force cut enters into the expression of the work with the − sign. For the curve BDA the motion is towards the left, and the flow of force cut is taken with the + sign.

If we agree to give the sign + to the flow of force cut by the circuit when the motion is towards the left of the observer, and the − sign when it is to the right, we may say that the total work is equal to the algebraical sum of the flow of force cut by the current.

458. Electromagnetic Action on a Current Element.— We are thus led to consider the action exerted on a current as resulting from the actions which would be exerted on each of the elements into which we may suppose it to be decomposed; it is the same problem as for a magnetic shell (344). To apply the result obtained to currents, we must replace the magnetic power of the shell by the strength of the current, and the force exerted on each element is expressed by

$$\text{(8)} \qquad \mathrm{IF}ds\sin\alpha = \mathrm{I}d\mathrm{A},$$

dA being the area of the parallelogram constructed on Fds. Thus:

The action exerted on a current element placed in a magnetic field, is equal to the product of the intensity of the current into the area of the parallelogram, drawn on a right line which represents the intensity of the field, and on the current element. This force is perpendicular to the parallelogram, and is directed to the left of the observer placed in the current, and who is looking in the direction of the force.

If the field is due to a single pole of mass m placed at P (Fig. 103), at a distance r from the element, we have $\mathrm{F}=\frac{m}{r^2}$; it follows that the reciprocal action of a current element and of a pole is expressed by

$$\text{(9)} \qquad d\phi = \frac{m\mathrm{I}}{r^2}ds\sin\alpha.$$

The action is therefore inversely as the square of the distance of the pole from the element; it is applied to the element, and is perpendicular to the plane passing through the element and the pole.

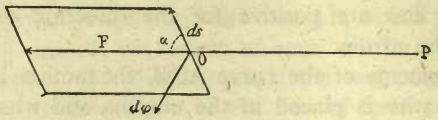

Fig. 103.

459. Reciprocal Action of Two Elements of a Current. —We have seen (347) that the action of two shells may be expressed as a function of the two contours. The action of the two currents may then be considered as the resultant of the actions exerted between the elements of the current which constitute it.

This elementary action $d^2\phi$ is not determinate, but if we assume that it takes place along the right line which joins the two elements, it is expressed by

$$d^2\psi = -\frac{4\Pi\Pi' ds ds'}{\sqrt{r}} \frac{\partial^2 \sqrt{r}}{\partial s \partial s'}.$$

If θ and θ' are the angles of the two elements with the right line joining them, and ϵ the angle which the two elements make with each other, we have

$$(10) \qquad d^2\psi = +\frac{2\Pi\Pi'}{r^2}\left\{\cos\epsilon - \frac{3}{2}\cos\theta\cos\theta'\right\} ds ds'.$$

The formulæ (9) and (10) represent the elementary laws discovered by Ampère.

The method adopted by Ampère to arrive at this result was entirely different; it will form the object of the following chapter.

460. Electromagnetic Intensity of a Current.—We have hitherto defined the intensity of the current by the quantity of electricity which passes through a section of the circuit in every unit of time. The strength thus defined is called the *electrostatic intensity;* it may be determined by measurements of capacities and potentials, or by electrochemical phenomena. The electromagnetic intensity introduced above (450) is defined by the condition of being ex-

pressed by the same number as the magnetic power of the equivalent shell of the same contour. We deduce from it

$$(11) \qquad I = \frac{ki}{2}.$$

We shall see later what is the significance of this factor $\frac{k}{2}$.

With the new expression for the intensity, the action of an unlimited rectilinear current at the distance a becomes

$$(12) \qquad \phi = \frac{2I}{a}.$$

Consequently, *the electromagnetic intensity equal to unity is that of the unlimited rectilinear current which at unit distance exerts a magnetic force equal to* 2.

461. ELECTROMAGNETIC UNITS.—The change in the expression for the strength necessarily leads to corresponding modifications in estimating other electrical quantities. If it is desired that the intensity shall always represent the quantity of electricity which traverses a section of a conductor in unit time, the equation

$$Q = It$$

will define Q, and therefore the unit of electricity.

If the current produces no other work than that of heating the circuit, Joule's law will define the expression of the resistance, and therefore the unit of resistance, by the ratio

$$W = I^2Rt,$$

in which W represents the thermal energy produced during the time t.

The electromotive force, lastly, is given by the equation

$$W = EIt.$$

The units thus defined, and which we shall call *electromagnetic units*, are those used in the following chapters. We shall subsequently establish the relations between them and the *electrostatic units*.

CHAPTER II.

ELEMENTARY ACTIONS.

462. AMPÈRE'S METHOD.—The course we have followed is, so to say, the inverse of that which led Ampère to the law of elementary actions. The importance of the subject, and the interest which Ampère's experiments and reasonings present, will justify the fresh statement of the question which we shall make, based on the ideas of this illustrious philosopher.

Ampère considered the actions exerted by currents, on magnets or on currents, as the resultant of the actions of each of the elements of length, into which the current may be decomposed, and he endeavoured to deduce from experiment the law of these elementary actions.

If we inquire to what extent an elementary law thus defined is directly accessible to experiment, it will be seen that in strictness it is possible to study the action of a single pole on a current element by working with a solenoidal magnet so long that the action of the other pole may be neglected, and with a portion of the current as small as we like, which is made movable; this, however, is not the case when we consider the action of a current element on a pole, or the interaction of two current elements. Only the entire circuit of the current, or in all cases a *closed current*, can be made to act on a movable current element as on a pole.

The investigation of a mathematical law, in the way in which Ampère views the problem, corresponds then—at least in the second case—to a purely mathematical conception; but the method is none the less legitimate so long as we merely propose to determine the resultant action of the whole circuit, the elementary law being then restricted only by the condition that the integral relative to a closed circuit gives a result which agrees with experiment. But it is clear, also, that the problem thus stated is not completely determinate, and that there may be several elementary laws which satisfy this fundamental condition.

463. Action of a Pole on a Current Element.—Fundamental Principles.—We will start from the following principles, some of which may be regarded as evident axioms, and others as experimental facts:—

I. *Equality of action and reaction.*—The action of a magnet on a current is equal and directly opposite to the action of the current on the magnet. This general law of Nature is experimentally verified in the present case; for if the magnet and the current are connected, the system if made free does not move.

II. *The action changes its sign with the sign of the pole and with the direction of the current.*—This fact is a result of experiment. The action remains the same when the sign of the pole and the direction of the current are simultaneously changed.

III. *Principle of sinuous currents.*—The action of a sinuous current on a magnet is identical with that of a rectilinear current which has the same terminals.

In order to verify this principle, Ampère showed that two conducting wires terminating at the same ends, one straight and the other sinuous, have no action on any magnet when they are traversed by the same current in opposite directions.

Some limitations are here necessary; the sinuous current must be of the same order of magnitude as the rectilinear current, and be but little distant from it; nor must it turn about the rectilinear current. This principle, moreover, will only be used to replace an element by its three projections.

IV. *The action of any given magnet, and therefore of a pole, on a current element, is perpendicular to the element.*

Ampère established this principle in the following manner. A metallic arc of a circle, movable about an axis passing through its centre, and perpendicular to its plane, can glide on two drops of mercury by which the current traversing it enters and leaves. Any given magnet placed in the vicinity leaves the arc at rest. The action of the magnet is then in the plane which passes through the axis of rotation, and is therefore perpendicular to the movable current. The arc, moreover, begins to move when the axis no longer passes through the centre.

V. *The action of a magnet on an element of current is applied to the element.*—This results from the following experiment, due to M. Liouville. Part of the rectilinear current is made movable about its axis; with this object, its ends dip in two small mercury cups by which the current enters. The rectilinear element does not rotate at all, in whatever manner the magnet is presented to it.

VI. *Principle of symmetry.*—The application of the principle of symmetry will determine the direction of the force.

We see at first that:

1st. The action of a pole on an element of current perpendicular to the right line which joins it to the pole, is perpendicular to the

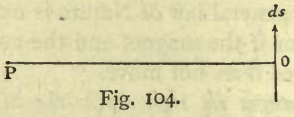

Fig. 104.

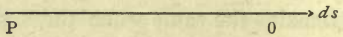

Fig. 105.

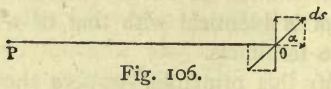

Fig. 106.

plane passing through the pole and the element. Let us join the pole P to the centre of the element *ds* (Fig. 104). We already know that the action is perpendicular to the element. It is also perpendicular to the right line PO, for if the figure is turned through 180° about the right line, the force must change its sign without changing direction (II).

2nd. The action of a pole on a current element, the prolongation of which passes through the pole, is zero. This action must be perpendicular to the element *ds* (Fig. 105); on the other hand, it should not change in direction when the element is made to turn by any quantity about the right line PO; it is therefore null.

Let there now be an element *ds* (Fig. 106) which makes an angle α with the right line joining it to the pole; the element of current *ds* may be replaced by its two projections $ds \cos\alpha$ and $ds \sin\alpha$, the one along the right line PO, the other in a perpendicular direction.

The action of the pole on the former is null; there only then remains the action of the pole on $ds \sin\alpha$. This latter is proportional, as we have seen, to the mass m of the pole, to the intensity i of the current; it is also proportional to the length $ds \sin\alpha$ of the element, and lastly to a certain function of the distance $f(r)$. Hence, if $d\phi$ is this force, and k a coefficient to be determined by experiment,

$$d\phi = mkids \sin\alpha f(r).$$

The force is moreover applied to the element, and is perpendicular to the plane P*ds*. Its direction is on the right of the current—that is, on the right of an observer placed in the element, and looking at the pole, for the action of the element on the pole is in the opposite direction.

VII. *Law of Biot and Savart.*—The experiments of Biot and Savart (444) demonstrate that the magnetic action of a rectilinear current on a pole, is inversely as the distance of the current from the pole.

According to a remark of Laplace, this law is satisfied if we assume that *the action of a pole on an element of current is inversely as the square of the distance*—that is to say, if we have $f(r)=\frac{1}{r^2}$. We may conversely prove that the law of the square is the only one which satisfies Biot and Savart's experiments.

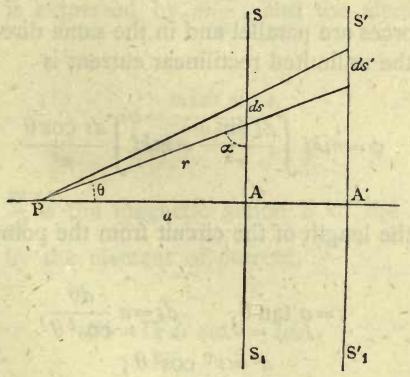

Fig. 107.

Consider, in fact, two parallel rectilinear currents, unlimited and of the same strength AS and A′S′, at distances a and a' from the pole P (Fig. 107). For two elements ds and ds', comprised between the same two radius vectors drawn through the point P, and the distances of which from this point are r and r', we have

$$\frac{ds}{ds'}=\frac{r}{r'}=\frac{a}{a'},$$

and therefore

$$rds'=r'ds$$

The ratio of the actions $d\phi$ and $d\phi'$ of the pole on the elements ds and ds' becomes then

$$\frac{d\phi}{d\phi'} = \frac{ds \sin\alpha \frac{1}{r^2}}{ds' \sin\alpha \frac{1}{r'^2}} = \frac{r'.r'\,ds}{r.rds'} = \frac{r'}{r} = \frac{a'}{a}.$$

The actions of the corresponding elements being inversely as the distances a and a', this will also be the case with the resultants. This is the law resulting from experiment. The action of a pole on an element of current is then expressed by

$$d\phi = mki \frac{ds \sin\alpha}{r^2}.$$

As all the forces are parallel and in the same direction, the action of the pole on the unlimited rectilinear current is

$$\phi = mki \int \frac{ds \sin\alpha}{r^2} = mki \int \frac{ds \cos\theta}{r^2}.$$

Measuring the length of the circuit from the point A, we have

$$s = a \tan\theta, \qquad ds = a \frac{d\theta}{\cos^2\theta},$$
$$a^2 = r^2 \cos^2\theta;$$

from which follows

$$\frac{ds \cos\theta}{r^2} = \frac{1}{a} \cos\theta\, d\theta,$$

and therefore

$$\phi = \frac{mki}{a} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\theta\, d\theta = \frac{2mki}{a}.$$

This force is by symmetry applied at the point A, and the action of the rectilinear current on the pole is applied at the same point, but in the opposite direction.

This result seems at first to disagree with experiment, for the action of the current on the pole is applied to the pole itself. This contradiction arises from the fact that in practice the current is necessarily closed. If, for simplicity, we suppose that the general circuit is in a plane passing through the point P, the actions $d\phi$ and $d\phi'$ of two corresponding elements ds and ds' situate at the angle $d\theta$, are in opposite directions, and inversely as the distances r and r'. The portion which closes the circuit being supposed to be very distant, the difference of the two forces is sensibly equal to the action of the element ds; but as $rd\phi = r'd\phi'$, the point of application of the partial resultant is the pole P. This is also the case for the general resultant. The action of the whole circuit is sensibly equal to that of the rectilinear part.

If the intensity is expressed by means of the electromagnetic unit (460), the action of the unlimited current on the pole m, placed at the distance a, is expressed by $m\frac{2I}{a}$, and the elementary formula becomes

$$(1) \qquad d\phi = \frac{mIds \sin \alpha}{r^2},$$

or, noting that $\frac{m}{r^2}$ is the magnetic action F of the mass m at the point occupied by the element of current,

$$(2) \qquad d\phi = IFds \sin \alpha = Id\mathrm{A},$$

dA denoting the surface of the parallelogram constructed on the element and on the force F.

The action exerted on the current Ids, situate in a magnetic field, only depends on the intensity of the field at this point, whatever be the system from which the force proceeds (458):

The action exerted on an element of current placed in a magnetic field is equal to the product of the intensity of the current by the area of the parallelogram constructed on the element of current, and on the intensity of the field. This force is perpendicular to the plane of the parallelogram, and directed to the left of the observer placed in the current who is looking in the direction of the field.

The plane of the parallelogram to which the magnetic force is perpendicular, was called by Ampère the *directive plane.*

Although we have given the name elementary to the force which we have defined, it cannot so be considered in the strict sense of the word; thus, as Ampère observes, "we cannot apply the term elementary either to a force which is manifested between two elements which are not of the same kind, or to a force which does not act along the straight line joining the two points between which it is exerted."

464. RECIPROCAL ACTION OF A POLE AND OF A CURRENT.— Starting from this elementary law we shall prove as above (346) that the components of the action of a unit pole, placed at the origin of the co-ordinates, on a current element ds situated at a point whose co-ordinates are x, y, and z, are

$$(3)\qquad \begin{aligned} d\xi &= \frac{I}{r^3}(ydz - zdy),\\ d\eta &= \frac{I}{r^3}(zdx - xdz),\\ d\zeta &= \frac{I}{r^3}(xdy - ydx). \end{aligned}$$

It may be observed that the moment $d\mathrm{M}_z$ of this force, in reference to the z axis, is

$$d\mathrm{M}_z = xd\eta - yd\xi = \frac{I}{r^3}\left[z(xdx + ydy) - (x^2 + y^2)dz\right].$$

The equation

$$x^2 + y^2 + z^2 = r^2$$

gives

$$xdx + ydy + zdz = rdr.$$

It follows that

$$d\mathrm{M}_z = \frac{I}{r^3}\left[z(rdr - zdz) - (r^2 - z^2)dz\right] = \frac{I}{r^2}(zdr - rdz) = -Id\left(\frac{z}{r}\right).$$

But $\frac{z}{r}$ is the cosine of the angle, which the right line r makes with the z axis; we have then

$$d\mathrm{M}_z = -Id\cos\gamma,$$

so that the moment M_2 of the actions exerted by the pole on any arc AB has the value

$$M_z = I(\cos\gamma_a - \cos\gamma_b). \tag{4}$$

If the circuit is closed, this moment is null, and as the direction of the z axis has been arbitrarily chosen, we see that *the action of a pole on a closed current passes through the pole.* Conversely, *the action of a closed current on a pole also passes through the pole.*

465. Instead of following the course taken, and of proving that Biot and Savart's law is satisfied by an action which is inversely as the square of the distance, we might have pursued a perhaps more rigorous course, and have admitted as an experimental fact that the action of a closed current on a pole passes through the pole.

The moment of the action of a pole on the element ds in reference to the z axis will then be

$$dM_z = -Ir^2 f(r)\, d\cos\gamma,$$

and the moment relative to an arc AB

$$M_z = -I\int_A^B r^2 f(r)\, d\cos\gamma = I\int_B^A r^2 f(r)\, d\cos\gamma.$$

Integrating this expression by parts, we get

$$M_z = I[r^2 f(r)\cos\gamma]_B^A - I\int_B^A \cos\gamma\, d[r^2 f(r)].$$

If the current is closed, the first term of the second member is null. As the moment must be null, whatever be the shape of the circuit traversed by the current, the second term must be identical with zero; that is to say, the product $r^2 f(r)$ must be a constant, and therefore the force must be inversely as the square of the distance.

If the arc, without being closed, terminates at two points, A and B of a right line, about which it can turn, the couple of rotation will not in general be zero.

Supposing, for instance, that the points A and B are situate on the same right line which passes through the pole, and on the same side of the pole, the moment of the forces in reference to this axis is null, and the current will not acquire any moment of rotation about the right line.

On the other hand, if the points A and B are on different sides of the pole, the angles γ_b and γ_a are equal, one to zero and the other to π: in this case the couple of rotation will be equal to $2I$, and the axis will turn continuously in the same direction.

We thus again find an explanation of the various special features in Faraday's experiment (456).

466. The components X, Y, and Z of the action of a current on a pole, placed at the origin of the co-ordinates, are, from what has been said,

$$\text{(5)} \qquad \begin{aligned} X &= -I\int\frac{ydz-zdy}{r^3}, \\ Y &= -I\int\frac{zdx-xdz}{r^3}, \\ Z &= -I\int\frac{xdy-ydx}{r^3}. \end{aligned}$$

If we put

$$\text{(6)} \qquad \begin{aligned} A &= \int\frac{ydz-zdy}{r^3} = G\cos\lambda, \\ B &= \int\frac{zdx-xdz}{r^3} = G\cos\mu, \\ C &= \int\frac{xdy-ydx}{r^3} = G\cos\gamma, \end{aligned}$$

with the condition

$$A^2+B^2+C^2=G^2,$$

we shall have

$$\begin{aligned} X &= -IA = -IG\cos\lambda, \\ Y &= -IB = -IG\cos\mu, \\ Z &= -IC = -IG\cos\nu, \end{aligned}$$

and therefore

$$\text{(7)} \qquad F=\sqrt{X^2+Y^2+Z^2}=IG.$$

The factor G is the action at the point P of the circuit, when it is traversed by a current of unit strength. We may represent this action by a straight line PG proportional to G, and making angles λ, μ, ν with the axes.

467. Equivalence of a Current and a Magnetic Shell.—The action of a magnetic field on a current element being identical with that of the same field on the corresponding element of the edge of a shell bounded by the current, it follows that the action of the current on a pole is identical with that of a shell of strength I, the positive face of which is on the left of the current.

The magnetic potential of a current at a point P is then, to within a constant, equal to the product of the strength I by the angle ω, under which we see from this point the positive side of a surface bounded by the circuit—that is, the face on the left of an observer who is going with the current, and looking towards the interior. The angle ω represents also the flow of force which a mass equal to unity, placed at the point P, would send towards this surface.

As the components of the force are equal and of opposite signs to the partial differentials of the potential, we see that the solid angle corresponding to a surface $d\omega$, seen from the origin of the co-ordinates, is given as a function of the contour by the equations

$$(8)\qquad \begin{aligned} \frac{\partial\omega}{\partial x} &= \int \frac{ydz - zdy}{r^3}, \\ \frac{\partial\omega}{\partial y} &= \int \frac{zdx - xdz}{r^3}, \\ \frac{\partial\omega}{\partial z} &= \int \frac{xdy - ydx}{r^3}. \end{aligned}$$

468. Action of Two Elements of a Current.—The action of two elements of a current may be established by an analogous method by the aid of some principles and of facts taken from experiment.

I. *Equality of action and reaction.*—This principle does not allow of experimental verification in the case of two elements of currents. It must be regarded as a fundamental hypothesis; it carries with it the necessary consequence that the action of two elements is along the right line which joins them. On the other hand, the reciprocal action of two elements of current is obviously

proportional to the length of each element, to the intensity of the current in each of them, and to a function, which remains to be determined, of the distance of the elements as well as of their relative distances.

II. *The action changes its direction when the direction of one of the currents is changed; it remains unaltered when the direction of the two currents is simultaneously changed.* This is a general property of electrical currents.

III. *Principle of symmetry.*—It follows from this principle of symmetry that the reciprocal action of two elements *a* and *b* (Fig. 108), one of which *a* is in the plane perpendicular to the other in its middle, is null.

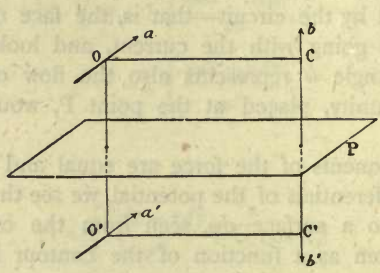

Fig. 108.

For consider a system *a'b'* symmetrical with the first in reference to a plane P parallel with the element *a*, and with the right line OC joining the centres of the elements.

The actions of *a* on *b*, and of *a'* on *b'* are respectively along OC and O'C', and in the same direction from symmetry. But the second is none other than the first, in which the direction of the current has been changed in the element *b;* the force should have changed its direction owing to this inversion, hence it is null.

The force in particular is null, if the element *a* is perpendicular to the right line OC, which joins the centres of the two elements, or directed along this right line. These are two cases which will have to be made use of.

IV. *Principle of sinuous currents.*—The principle of sinuous currents may be applied as above (463) and with the same limitations; we can always replace a current element by its projections on three rectangular axes.

Let us consider two elements a and b (Fig. 109) in any position; let ds and ds' be their lengths, i and i' the intensities of the two currents referred to a given unit, θ and θ' the angles of their directions with the right line OO′ joining their centres, r the distance OO′, lastly ω the angle of the planes drawn through the right line OO′ and the two elements.

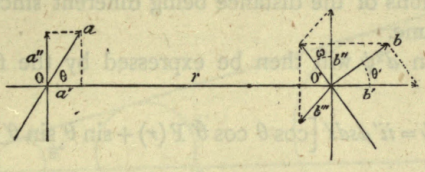

Fig. 109.

Let us take the plane which passes through the element ds and the right line OO′ as plane of the figure, and replace each of these elements by its projections on three rectangular axes; one of these axes is the right line OO′, the other a right line in the plane of the figure, and the third a perpendicular to this plane. The element a has only two projections

$$a' = ds \cos\theta,$$
$$a'' = ds \sin\theta;$$

the three projections of the element b are

$$b' = ds' \cos\theta',$$
$$b'' = ds' \sin\theta' \cos\omega,$$
$$b''' = ds' \cos\theta' \sin\omega.$$

The total action consists of the actions of each of these elements a' and a'' on each of the elements b', b'', and b'''.

Of these six actions, four are null from the principle of symmetry, that of a' on b'' and b''', and that of a'' on b' and b'''.

There only remains to be examined the action of a' on b', and that of a'' on b''.

The former is exerted between two elements directed along the same right line; it might be represented by

$$ii'dsds \cos\theta \cos\theta' \mathrm{F}(r).$$

The second is exerted between two elements parallel to each other, and perpendicular to the right line joining their centres; we might represent it by

$$ii'dsds' \sin\theta \sin\theta' \cos\omega f(r),$$

the two functions of the distance being different since the conditions are not the same.

The action $d^2\psi$ will then be expressed by the formula

$$(9) \qquad d^2\psi = ii'dsds' \left[\cos\theta \cos\theta' \mathrm{F}(r) + \sin\theta \sin\theta' \cos\omega f(r)\right].$$

If ϵ be the angle of the two elements, we have

$$\cos\epsilon = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\omega,$$

and we may write

$$(10) \qquad d^2\psi = ii'ds\,ds' \left[\cos\theta \cos\theta' \left[\mathrm{F}(r) - f(r)\right] + \cos\epsilon f(r)\right].$$

469. DETERMINATION OF THE FUNCTIONS F(r) and $f(r)$.—To determine the functions F(r) and $f(r)$, we must have recourse to experiment, and may employ very different methods, according to the phenomenon to which we apply ourselves. We shall adopt the following course, which is not perhaps the most rigorous from the mathematical point of view, but which leads most rapidly to the final formula.

We start from the two following experiments devised by Ampère.

V. *When the homologous dimensions of three similar currents of the same intensity are in geometrical progression—that is to say, as* 1, m, m^2, *and are moreover similarly placed—the actions of the extreme currents on the intermediate current are equal and of opposite sign.* If this latter is movable along a line passing through its centre of similitude, and if it is disturbed from its position of equilibrium, it returns to it of itself—that is to say, that the equilibrium is stable.

Ampère made this experiment with three circular currents situate in the same plane, the intermediate circuit being movable about an axis perpendicular to this plane.

VI. *The action of a closed current on an element of current is perpendicular to the element.*—The arrangement of this latter experiment is the same as that for the action of magnets on currents (463, IV.)

Consider the three similar currents of the first experiment (V.) For the position of equilibrium, the distances to the centre of

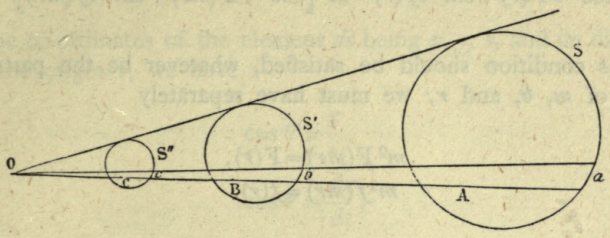

Fig. 110.

similitude of the three homologous points, A, B, C (Fig. 110) of the circles, satisfy the ratio

$$\frac{OA}{1}=\frac{OB}{m}=\frac{OC}{m^2};$$

from which is deduced

$$OA-OB=AB=OA(1-m),$$
$$OB-OC=BC=OA\,m(1-m),$$

and therefore

$$\frac{AB}{BC}=\frac{1}{m}.$$

For three homologous elements of current a, b, and c, the lengths will be ds, mds, and m^2ds; the distance of the two former being r, that of the second to the third will be mr.

If we assume that each intermediate element such as b is in equilibrium between the two others a and c, which correspond to it, the entire current S' will be in equilibrium between the two similarly placed currents S and S''. It does not seem that this condition is always necessary, but it is evidently sufficient, and it enables us to determine the form of the two functions F(r) and $f(r)$.

It follows, in fact, that the action exerted on the element b ought not to change when a is replaced by c—that is to say, ds by m^2ds, and r by mr; we get then, from equation (9), suppressing the common factor $ii'ds\,ds'$, and observing that the angles θ and θ' are equal, and the angle ω is zero,

$$\cos^2\theta\,\mathrm{F}(r)+\sin^2\theta f(r)=m^2\left[\cos^2\theta\,\mathrm{F}(mr)+\sin^2\theta f(mr)\right].$$

This condition should be satisfied, whatever be the particular values of m, θ, and r; we must have separately

$$m^2\,\mathrm{F}(mr)=\mathrm{F}(r),$$
$$m^2 f(mr)=f(r).$$

Making $r=1$, and $m=r$, we get

$$r^2\,\mathrm{F}(r)=\text{const}=h,$$
$$r^2 f(r)=\text{const}=kh,$$

and, consequently,

$$\mathrm{F}(r)=\frac{h}{r^2},$$

$$f(r)=k\frac{h}{r^2}.$$

Thus the functions $\mathrm{F}(r)$ and $f(r)$ are both inversely as the square of the distance.

The expression for the elementary action then becomes

$$(11)\qquad d^2\psi=\frac{hii'dsds'}{r^2}\left[k\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\omega\right],$$

or

$$(12)\qquad d^2\psi=\frac{hii'dsds'}{r^2}\left[(k-1)\cos\theta\cos\theta'+\cos\epsilon\right].$$

470. DETERMINATION OF THE RATIO OF THE TWO CONSTANTS.—The last experiment (VI.) enables us to determine the ratio k of the two constants.

Let us place the origin of the co-ordinates at the centre of the movable element ds', and the x axis in the direction of the element itself. The action of an element ds of a closed circuit in which the intensity is i is expressed, as we have seen, by

$$d^2\psi = \frac{hii'dsds'}{r^2}\left[(k-1)\cos\theta\cos\theta' + \cos\epsilon\right].$$

The co-ordinates of the element ds being x, y, z, and its distance from the origin r, we have

$$\cos\theta' = \frac{x}{r},$$

$$\cos\theta = \frac{dr}{ds},$$

$$\cos\epsilon = \frac{dx}{ds}.$$

The elementary action may then be written in the form

$$d^2\psi = \frac{hii'dsds'}{r^2}\left[(k-1)\frac{x}{r}\frac{dr}{ds} + \frac{dx}{ds}\right]$$

$$= hii'ds'\left[(k-1)\frac{x}{r^3}dr + \frac{dx}{r^2}\right],$$

and the projection of this force on the x axis is

$$d^2\psi\cos\theta' = d^2\psi\frac{x}{r} = hii'ds'\left[(k-1)\frac{x^2dr}{r^4} + \frac{xdx}{r^3}\right].$$

The component parallel to the x axis of the action of the closed circuit on the element ds' is expressed by

$$d\xi' = \int d^2\psi\frac{x}{r} = hii'ds'\left[(k-1)\int\frac{x^2dr}{r^4} + \int\frac{xdx}{r^3}\right].$$

Integration by parts gives

$$\int x^2r^{-4}dr = \left[-\frac{1}{3}r^{-3}x^2\right] + \frac{2}{3}\int\frac{xdx}{r^3}$$

For a closed circuit, the first term of the second member is null; we get then

$$d\xi' = hii'ds'\left[\frac{2}{3}(k-1)+1\right]\int\frac{xdx}{r^3} = \frac{hii'ds'}{3}(2k+1)\int\frac{xdx}{r^3}.$$

Since from experiment this component must be null, we get

$$2k+1=0, \qquad \text{or} \qquad k=-\frac{1}{2}.$$

With this value of k, the elementary action becomes

$$(13) \qquad d^2\psi = h\frac{ii'ds'ds}{r^2}\left[\cos\epsilon - \frac{3}{2}\cos\theta\cos\theta'\right],$$

or

$$d^2\psi = hii'ds'\left[\frac{dx}{r^2} - \frac{3}{2}\frac{xdr}{r^3}\right].$$

471. DETERMINATION OF THE CONSTANT h.—The components of the action of the current parallel to the other axes are then

$$d\eta' = \int d^2\psi\frac{y}{x} = hii'ds'\int\left[\frac{ydx}{r^3} - \frac{3}{2}\frac{xydr}{r^4}\right],$$
$$d\zeta' = \int d^2\psi\frac{z}{r} = hii'ds'\int\left[\frac{zdx}{r^3} - \frac{3}{2}\frac{xzdr}{r^4}\right].$$

Integration by parts gives

$$\int\frac{xydr}{r^4} = \left(-\frac{1}{3}\frac{xy}{r^3}\right) + \frac{1}{3}\int\frac{xdy+ydx}{r^3}.$$

The former term of the second member being null for a closed circuit, we have lastly

$$(14) \qquad \begin{aligned} d\eta' &= \frac{hii'ds'}{2}\int\frac{ydx-xdy}{r^3}, \\ d\zeta' &= \frac{hii'ds'}{2}\int\frac{zdx-xdz}{r^3}. \end{aligned}$$

The value of the action F, of the current on unit magnetic mass placed at the origin, and therefore the intensity of the field which this current produces at the point where is the element, is expressed by IG, where I is the electromagnetic intensity of the current (466), and its components are

$$X = -IA = -I\int\frac{ydz - zdy}{r^3},$$

$$Y = -IB = -I\int\frac{zdx - xdz}{r^3},$$

$$Z = -IC = -I\int\frac{xdy - ydx}{r^3}.$$

The three components $d\xi'$, $d\eta'$, $d\zeta'$, of the action $d\phi'$ of the circuit on the element ds' may then be written

$$d\xi' = 0,$$

$$d\eta' = -\frac{hii'ds'}{2}C,$$

$$d\zeta' = +\frac{hii'ds'}{2}B,$$

from which is deduced

$$Xd\xi' + Yd\eta' + Zd\zeta' = 0.$$

It follows from this that the two forces F and $d\phi'$ are perpendicular to each other.

As the x axis is the only one which is defined, we may choose the two others in such a way that the magnetic action F of the current is in the plane xz; we have then

$$Y = 0, \qquad B = 0,$$
$$X = F\cos\alpha, \qquad A = G\cos\alpha,$$
$$Z = F\sin\alpha, \qquad C = G\sin\alpha,$$

α being the angle which the force F on the straight line G makes with the x axis.

It follows from this that

$$d\xi' = 0,$$
$$d\zeta' = 0,$$
$$d\eta' = -\frac{hii'ds'}{2} G \sin\alpha = -\frac{hii'ds'}{2I} F \sin\alpha = d\phi'.$$

The action of the closed circuit on the element is therefore perpendicular to the force F and to the element ds'—that is to say, Ampère's *directive plane*—and proportional to the surface of the parallelogram constructed on the force F, and the element ds'.

If the force of the field $F = IG$, at the point where is the element ds', was produced by a magnetic system, the action in like manner would be along the y axis, and its value would be $I'ds'F \sin\alpha$, where I' is the electromagnetic intensity of the current which traverses the element.

The two actions are in the same direction, and they are proportional; if we *assume* that they are identical, it will follow that

$$h = 2\frac{II'}{ii'}.$$

As the numerical expression of a magnitude is inversely as the unit with which it is measured, it will be seen that the constant h is equal to twice the square of the ratio arbitrarily chosen to measure the strength of the current in electromagnetic units.

472. If we suppose that the currents have, in the first place, been determined in electromagnetic units, we have then

$$h = 2;$$

we thus arrive at Ampère's formula, which we had already obtained (459),

$$(15)\quad d^2\psi = \frac{2II'dsds'}{r^2}\left[\sin\theta\sin\theta'\cos\omega - \frac{1}{2}\cos\theta\cos\theta'\right],$$

$$(16)\quad d^2\psi = \frac{2II'dsds'}{r^2}\left[\cos\epsilon - \frac{3}{2}\cos\theta\cos\theta'\right],$$

and which we may write in the more symmetrical form (351)

$$d^2\psi = \frac{4II'dsds'}{\sqrt{r}}\frac{d^2\sqrt{r}}{dsds'}.$$

473. ELECTRODYNAMIC UNIT OF INTENSITY.—If, with Ampère, we directly make $h = 1$ in formula (13), the strength of the current will be expressed as a function of a particular unit, which is called the *electrodynamic unit.*

This unit will be defined by the formula itself. By making

$$\theta = \theta' = \frac{\pi}{2},$$

$$\epsilon = 0,$$

$$ds = ds' = 1,$$

$$i = i' = 1,$$

we get

$$d^2\psi = 1.$$

In this case the currents are parallel, of unit length, perpendicular to the line which joins their centres, and at unit distance; the strength of the current, which is equal for each of them, and is taken at unity, is such that the reciprocal action is equal to the unit of force.

Supposing the currents equal, equation (14) will give,

$$i^2 = 2\mathrm{I}^2, \qquad i = \mathrm{I}\sqrt{2}.$$

The electrodynamic intensity of a current is equal to its electromagnetic intensity multiplied by $\sqrt{2}$.

In virtue of the ratio which connects the numerical expression of a magnitude into the unit which serves to measure it, we see that *the electrodynamic unit of current is equal to the electromagnetic unit divided by* $\sqrt{2}$.

474. The identity between the mutual action of currents and that of the correlated magnetic systems has been confirmed in all experiments—as long, at least, as a steady condition has been established in the circuits.

We may cite, for instance, the experiments of Weber on the reciprocal action of the cylindrical coils with circular bases. This action is proportional to the strength of the two currents; it varies with the relative distance and direction of the coils according to the same law as that of two magnets whose axes are respectively parallel to the axes of the coils.

475. FORMULÆ EQUIVALENT TO THAT OF AMPÈRE.—We have seen (349) that the action of two elements of the contour of two magnetic shells, which is equivalent to the elementary electrodynamic action, may be expressed in an infinity of different ways, with this condition that the resultant of the actions of a closed circuit on an element has a determinate value.

476. (I.) *Formula of M. Reynard.*—The first form which we have met (348) for the action of ds upon ds' is, by supposing the element ds' at the origin of co-ordinates and directed along the x axis, a force whose components are

$$f_x = 0,$$

$$f_y = -a\frac{x^2}{r^3}d\frac{y}{x} = a\frac{y^2}{r^3}d\frac{x}{y},$$

$$f_z = -a\frac{x^2}{r^3}d\frac{z}{x} = a\frac{z^2}{r^3}d\frac{x}{z}.$$

The factor a in these equations represents the product $\Pi' ds'$, and x, y, z are the co-ordinates of the element ds.

The force itself is expressed by the formula

$$f = \frac{\Pi' ds' ds}{r^2}\sin\theta\cos\mu',$$

in which θ is the angle of the element ds with the right line r, and μ' the angle of the element ds' with the plane rds.

If $d\beta$ is the angle under which the element ds is seen from the element ds', an angle which is equal to $\frac{ds\sin\theta}{r}$, this formula may still be written

$$f = \frac{\Pi' ds'}{r}\cos\mu' d\beta,$$

which is the formula of M. Reynard.

In order to determine the direction of this elementary force, we observe, in the first place, that it is perpendicular to the element ds' since $f_x = 0$.

It is in the plane rds. The equation of this plane, of which X, Y, and Z are the co-ordinates, is

$$X(ydz - zdy) + Y(zdx - ydz) + Z(xdy - ydx) = 0.$$

The intersection with the yz plane is

$$Y(zdx - xdz) + Z(xdy - ydx) = 0 ;$$

from which follows

$$\frac{Y}{f_y} = \frac{Z}{f_z}.$$

We have further

$$r^2 ds^2 \sin^2\theta = (ydz - zdy)^2 + (zdx - xdz)^2 + (xdy - ydx)^2$$

$$= r^6 \frac{(f_y^2 + f_z^2)}{a^2} + (ydz - zdy)^2 = \frac{r^6 f^2}{a^2} + (ydz - zdy)^2,$$

which gives

$$f^2 = \frac{a^2 ds^2 \sin^2\theta}{r^4}\left[1 - \frac{(ydz - zdy)^2}{r^2 ds^2 \sin^2\theta}\right].$$

Now, the expression $\dfrac{ydz - zdy}{rds \sin\theta}$ is the cosine of the angle which the perpendicular to the plane rds makes with the x axis : hence the quantity in brackets is the square of the sine of this angle, or the square of the cosine of the angle μ' which the plane makes with the x axis—that is to say, with the element ds', and we have

$$f = \frac{ads \sin\theta \cos\mu'}{r^2} = \frac{\text{II}' ds ds'}{r^2} \sin\theta \cos\mu'.$$

Thus the action of ds upon ds' is in the plane rds, perpendicular to the element ds', proportional to the sine of the angle which the element ds makes with the distance r, and to the cosine of the angle which the element ds' makes with the plane rds, and lastly inversely as the square of the distance.

Let us take the plane rds as that of xz, and in this plane the line OO′, which joins the two elements, as the x axis. The force acting on the element ds' placed at the origin of the co-ordinates is in the xz plane and is perpendicular to ds'. To obtain its direction, we must project the element ds' on the xz plane; a straight line in this plane, perpendicular to the projection, will be the direction in question; it is perpendicular to the projecting plane, and therefore to the element which passes through its foot in this plane.

The components of this force parallel to the axes are

$$f_x = f\cos\beta = \frac{\Pi' ds \sin\theta}{x^2} ds' \cos\mu' \cos\beta = \frac{\Pi' dzdz'}{x^2},$$

$$f_y = 0,$$

$$f_z = f\sin\beta = \frac{\Pi' ds \sin\theta}{x^2} ds' \cos\mu' \sin\beta = \frac{\Pi' dzdx'}{x^2}.$$

If we still denote by θ' the angle which the right line r makes with the element ds', and by ω the angle of the two planes rds and rds', we have

$$dz' = ds' \sin\theta' \cos\omega,$$
$$dx' = ds' \cos\theta',$$

which gives

$$f_x = \frac{\Pi'}{r^2} \sin\theta \sin\theta' \cos\omega \, dsds',$$

$$f_z = \frac{\Pi'}{r^2} \sin\theta \cos\theta' \, dsds'.$$

The action of two elements of consecutive currents is evidently null.

In fine, we have not here an equal and opposite action and reaction, but there is a different action on each of the two elements, directed perpendicularly to this element, and in the plane determined by the other element.

The existence of a force perpendicular to the element is incompatible with the idea of an action at a distance; but if, on the contrary, we view electrodynamic forces as resulting from a modification in the elastic properties of the medium, we can easily see that the reaction of this medium on an element of current may be perpendicular.

477. (II.) *General Formula.*—We may add an exact differential of the co-ordinates to each of the components f_x, f_y, and f_z without the action of the closed circuit on the element ds' being modified. We may then take as components of the elementary action the following general expressions in which X, Y, and Z, are any given functions of the co-ordinates:

$$d^2\xi' = a dX,$$

$$d^2\eta' = a\left[dY - \frac{x^2}{r^3} d\frac{y}{x}\right],$$

$$d^2\zeta' = a\left[dZ - \frac{x^2}{r^3} d\frac{z}{x}\right],$$

478. (III.) *Ampère's Formula.*—If we impose on the elementary force the condition of being directed along the right line joining the two elements, we get Ampère's formula; this formula is the only one which satisfies the general principle of action and reaction, and consequently the essential conditions of a true elementary force. For any other solution, the action of the element ds on the element ds', will not be equal and directly opposite that of the element ds' on the element ds.

479. (IV.) *Grassmann's Formula.*—Let us replace the arbitrary functions X, Y, and Z, respectively by xf, yf, and zf, f denoting a function of the distance r. The components of the elementary force will then be

$$d^2\xi' = ad(xf),$$
$$d^2\eta' = a\left[d(yf) - \frac{x^2}{r^3}d\frac{y}{x}\right].$$
$$d^2\zeta' = a\left[d(zf) - \frac{x^2}{r^3}d\frac{z}{x}\right].$$

This operation is the same as adding to the force given by M. Reynard's formula, another force $d^2\psi_1$, the components of which are

$$d^2\xi_1 = a\,d(xf) = a[fdx + xdf],$$
$$d^2\eta_1 = a\,d(yf) = a[fdy + ydf],$$
$$d^2\xi_2 = a\,d(zf) = a[fdz + zdf],$$

The force itself is given by the equation

$$[d^2\psi_1]^2 = a^2[f^2ds^2 + r^2(df)^2 + 2frd(fdr)],$$

or, taking into account the relation $dr = ds\cos\theta$,

$$\frac{1}{a^2}(d^2\psi_1)^2 = [fdr + rdf]^2 + f^2ds^2\sin^2\theta = [d(rf)]^2 + f^2ds^2\sin^2\theta.$$

This force makes with the straight line r an angle, the cosine of which is

$$a\frac{xd(xf) + yd(yf) + zd(zf)}{rd^2\psi_1} = a\frac{frdr + r^2df}{rd^2\psi_1} = a\frac{d(rf)}{d^2\psi_1}.$$

Finally, the angle δ which it makes with the element ds is

$$\cos\delta = \frac{a}{dsd^2\psi_1}[dxd(xf) + dyd(yf) + dzd(zf)],$$

or

$$\cos\delta = \frac{adr}{dsd^2\psi_1}d(rf) + \frac{afds}{d^2\psi_1}\sin\theta.$$

If we impose on this force the condition of being perpendicular to the right line which joins the two elements we have then

$$d(rf) = 0,$$

from which is deduced

$$rf = \mathrm{A}, \qquad f = \frac{\mathrm{A}}{r};$$

hence the value of the added force is

$$d^2\psi_1 = afds\sin\theta = \frac{\mathrm{A}a}{r}ds\sin\theta.$$

It will moreover be seen that this force $d^2\psi_1$ makes with the element ds an angle equal to $\theta + \frac{\pi}{2}$.

When the two elements are directed along the same straight line, as the force $d^2\psi_1$ is null, and the force given by M. Reynard's formula is also null, the action of the two elements is null.

On this hypothesis, which is that of Grassmann, the true force $d^2\psi$ will be the resultant of a force which is inversely as the square of the distance, and of a force $d^2\psi_1$ which is inversely as the distance, is perpendicular to the right line joining the elements, and whose direction makes with the element ds an angle equal to $\frac{\pi}{2} + \theta$.

Several other conditions might be imagined equally compatible with experiment; but these few examples will suffice to show the indeterminateness of the problem, and to point out the principal solutions.

CHAPTER III.

PARTICULAR CASES.

480. ACTION OF TWO PARALLEL CURRENTS.—According to Ampère's formula, two elements of currents parallel to each other, and perpendicular to the right line which joins their centres, will attract or repel according as the currents are in the same, or in opposite directions.

This result is usually verified by bringing a rectilinear current, which we may suppose unlimited, near a portion of a rectilinear current movable parallel to itself. The experiment is really more complicated, for each of the currents in question forms part of a closed circuit. In whatever manner we may suppose the planes of the two currents placed in reference to each other, bringing the two rectilinear portions near each other will increase, for each of them, the flow of force which it will receive from the other by its negative surface, and will diminish the relative energy if the currents are in the same direction; the converse takes place when they are in opposite directions.

Let I be the intensity of the unlimited current, I′ that of the finite current, which is parallel to it, and b its length. If we vary the distance a of the two currents (which we suppose in the same direction) by da, the variation in the flow of force which enters the circuit from the movable current is

$$dQ = -bda\frac{2I}{a} = -2Ib\frac{da}{a};$$

the force exerted upon the movable part of the circuit is expressed by $-I'\frac{dQ}{da} = 2II'\frac{b}{a}$, it is therefore inversely as the distance a.

481. ANGULAR CURRENTS.—Two rectilinear currents placed near each other tend to set parallel. This result is usually enunciated

by saying, that two currents which form an angle with each other attract if they both approach, or both recede from the apex of the angle or the common perpendicular, and that in the contrary case they repel.

The experiment is made by bringing an unlimited rectilinear current near the bottom of a movable rectangular frame traversed by the current. The movable frame turns so as to receive, on its negative face, the maximum flow of force which proceeds from the rectilinear current. There is no simple expression for the work of any given displacement; but the total work corresponding to the displacement of the frame, from the position in which its plane is perpendicular to the current, to that in which it becomes parallel, is proportional to the flow of force which traverses the frame in the second case. If a_0 and a_1 are the distances from the unlimited current of the two sides of the frame parallel to it, and b the length of one of these sides, we have

$$Q = -2Ib\int_{a_0}^{a_1}\frac{da}{a} = -2Ibl.\frac{a_1}{a_0},$$

and the electromagnetic work is equal to $2II'bl.\frac{a_1}{a_0}$.

These movements are easily accounted for by supposing the currents replaced by equivalent magnetic shells, and considering the actions of these shells.

We may arrive at the same object on Faraday's plan, by considering the lines of force and their distribution in the field. The figured lines of force of the field resulting from the various systems near each other, are closer in certain regions than in others. If we represent these lines of force (105) as elastic threads exposed to a strain in the direction of their length, and to a repulsion in the direction perpendicular to this, we shall have a very definite idea of the relative motion which they tend to produce.

482. Apparent Repulsion of Two Consecutive Elements of Current.—This important experiment of Ampère consists in putting the two poles of a battery in connection with two rectangular troughs containing mercury, and separated by an insulating division. A copper wire is bent so as to form two horizontal legs floating on the mercury, and a cross piece (in the form of a bridge) which connects the two former. When the battery circuit is closed, the wire is seen to glide along the surface of the mercury, and to recede from the points by which the current enters.

Ampère thought this a proof that the two elements of current directed along the same right line, and in the same direction, repel each other, as the elementary formula indicates; but it is easy to see that the interpretation of the phenomena does not entail this consequence.

In this experiment the current traverses a circuit, one of whose portions is movable, and the surface of which tends to become a maximum (455). This result may moreover be arrived at directly by replacing the current by a flexible shell bent upon itself, as shown in Fig. 111. The three shells superposed in the space ABB'A' do not give rise to any force among them parallel to the plane of the current; but their external action is equivalent to that of a simple shell. The portion aDCb tends to recede, and the shell extends so as to occupy the greatest surface.

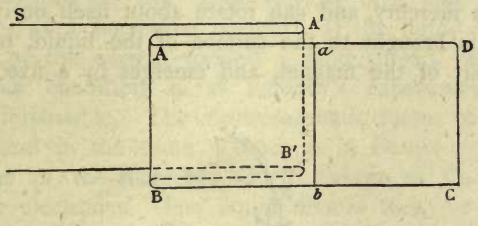

Fig. 111.

483. Electromagnetic Rotation.—1st. *Barlow's Wheel.*—A toothed metal wheel, movable about a horizontal axis, is arranged so that one or more teeth plunge with their lower ends in a cup containing mercury. If the system is traversed by a current which enters by the axis and leaves by the mercury, the only action of the current on itself will tend to move the bottom teeth in a direction which displaces them from the rest of the circuit so as to increase the total surface; but this action is generally too weak to overcome friction. A stronger effect is obtained by putting the trough between the limbs of a horseshoe magnet arranged horizontally. The lines of magnetic force then traverse the plane of the wheel; if they are directed from back to front—that is, with the north pole in front—the rotation will be in the opposite direction of the hands of a watch.

In order to get a phenomenon easy to calculate, replace the magnet by a uniform magnetic field of intensity F, parallel to the axis of the rotation. Let a be the radius of the wheel, θ the angle of

two consecutive teeth, and suppose the surface of the mercury placed so that one of the teeth touches the liquid at the moment the preceding one quits it. The flow of force through the triangle formed by the radius, which corresponds to these two teeth, is $F\frac{a^2 sin\theta}{2}$, or sensibly $F\frac{a^2\theta}{2}$; that is to say, the product of the force by the surface of the sector, and the corresponding work is $IF\frac{a^2\theta}{2}$. For an entire turn the work is IFS—that is to say, proportional to the whole surface S of the wheel.

484. 2nd. *Ampère's Experiment.*—This experiment, in which the rotation of a magnet is produced by a current, is the converse of that, the theory of which has been given above (456). The apparatus is arranged so that only one of the poles of the magnet can traverse the current; a continuous rotation is obtained in this way. The magnet (Fig. 112), loaded by a counterpoise of platinum, floats on the mercury, and can rotate about itself on its own axis; the current is brought to the surface of the liquid, traverses the projecting part of the magnet, and emerges by a fixed conductor

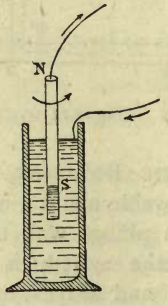

Fig. 112.

which dips in a drop of mercury in the top N. If we suppose that the current goes rigorously along the axis of the magnet, the work at each turn for a magnetic mass outside the axis is $4\pi m$I, and gives rise to a couple the moment of which is $2m$I. The phenomenon is really more complicated, because the current traverses the whole section of the magnet.

Faraday repeated the experiment by placing the magnet outside the circuit. The magnet is brought to the centre of the vessel by a metal rod, and the magnet floats in an eccentric position.

In both cases, if the current ascends by the axis, and the top of

the magnet is a north pole, the rotation is opposite to the motion of the hands of a watch. Faraday's arrangement gives greater friction, and the rotation is less rapid.

485. ROTATION OF LIQUIDS.—When the current traverses a liquid, the liquid filaments, which coincide with the lines of electrical flow, may be considered as movable circuits, capable of obeying electromagnetic actions, and experiment shows that the liquid is moved along with the current which it carries.

1st. *Davy's Experiment.*—Two platinum electrodes just project a very little below the top of the mercury. If the N pole of a magnet is placed over one of them—the negative electrode, for example—a depression of the mercury is observed; and, at the same time, a rotation in the same direction as the hands of a watch.

2nd. *M. Jamin's Experiment.*—The two electrodes of a voltameter are placed in the same vertical line, and on the axis of the poles of a horseshoe magnet. If the liquid molecules in a filament of the current formed a rigid thread, we should be in the same condition as in Faraday's experiment, in which rotation is impossible. The electromagnetic forces really act independently, and in the same manner as in Davy's experiment, on the portions of the filaments which diverge as they start from each of the electrodes. The liquid divides them into two superposed layers which rotate in contrary directions, and the rotation is made visible by the bubbles of gas which result from the decomposition of water.

3rd. *M. Bertin's Experiment.*—In M. Bertin's experiments the movement of the liquid is made visible by small pieces of cork which float on the surface. The liquid is in an annular dish containing two rings of metal, one inside the other. If these circles are electrodes, a series of either centripetal or centrifrigal radiating currents is obtained in the liquid. When a magnet is placed in the axis of the current, the liquid acquires a rotation in a definite direction in agreement with theory. The direction of the rotation is not altered, if the central magnet is replaced by a magnetised tube encircling the dish. For if the north pole is at the top in the two cases, the flow of magnetic force is diverted downwards, both inside and outside the hollow magnet. This is not the case when the magnet is replaced by a coil: the rotation of the liquid changes its direction according as the coil is inside or outside the dish, and each of the lines of force constitutes a closed circuit.

486. Electrodynamic Rotation. — Consider an unlimited rectilinear current X'X of strength I (Fig. 113), and a finite rectilinear current of length a, and strength I, perpendicular to the first

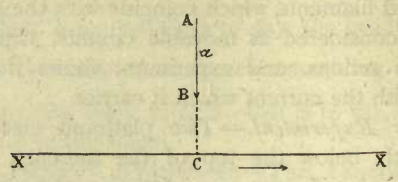

Fig. 113.

and in the same plane. If we give the current a a displacement dx parallel to the current I, then if r_0 is the distance BC, the corresponding work will be

$$2\Pi\Pi' dx \int_{r_0}^{r_0+a} \frac{dr}{r} = 2\Pi\Pi' dx l. \left(1 + \frac{a}{r_0}\right).$$

The force which acts on the movable current is perpendicular to its direction, parallel to the unlimited current, and its value is $2\Pi\Pi' l. \left(1 + \frac{a}{r_0}\right)$. This current will be carried parallel to itself by a constant force, and will ascend or descend the unlimited current according as it is ascending or descending in reference to this latter.

The experiment is ordinarily made by causing a circular current to act on a portion of a current movable about an axis perpendicular to its plane and passing through its centre. The movable current then rotates in the opposite direction to the principal current.

If the movable current is closed, or if its ends are on the axis, there is evidently no movement, for then each line of force meets the edge twice.

487. Action of a Uniform Field.—Consider first two unlimited rectilinear conductors AA', BB' (Fig. 114) parallel to each other at the distance b, and let us suppose that, while the two ends are in communication with the poles of a battery, the circuit is closed by a cross bar CC', movable parallel to itself along the conductors AA' and BB'.

Let Z be the component of the intensity of the field perpendicular to the plane of the conductors; for a displacement dx of

the movable bar, the variation in the flow of magnetic force in the circuit is $bZdx$. In the case of the terrestrial field, and if the rails are horizontal, the component Z is directed downwards in our hemisphere, and with the direction of the current shown by the arrows, the movable bridge CC′ will recede from AB under the influence of electromagnetic forces; it will approach, if the current is in the opposite direction.

In Ampère's experiment (482) the action of the current on itself tends to increase the surface, and to repel the movable bridge. This action and that of the Earth will add themselves or oppose according to the direction of the current; the motion of the wire is more or less easy according to the case.

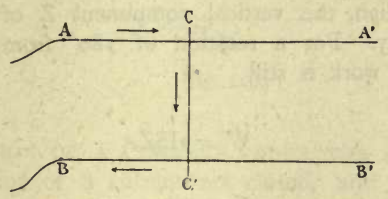

Fig. 114.

488. Suppose now that the movable conductor forms a closed circuit; let S be the surface of this circuit, if it is plane, or the maximum projection of its surface on a plane, which we will call the plane of the current. When such a circuit is movable in a uniform magnetic field, like that of the earth, stable equilibrium corresponds to the case in which the flow of force across the negative face of the circuit is a maximum; the plane of the circuit tends then to set at right angles to the force. Under the influence of the earth this plane will be perpendicular to the dipping needle; the current will move from east to west in the lower part. If F is the strength of the field, I that of the current, the potential energy of the current in the position of equilibrium is

$$W_1 = -ISF;$$

when the face of the current is turned upside down, it becomes

$$W_2 = ISF.$$

The work done against electromagnetic forces in this operation is then

$$W_2 - W_1 = 2ISF.$$

If the current is made to turn about a vertical axis we need only consider the horizontal component H of the field. The work of the rotation of 180° about the axis from the position of equilibrium is then

$$W' = 2ISH.$$

If the current turns about a horizontal axis parallel to the magnetic meridian, the vertical component Z of the field alone comes into play. For a rotation of 180° from the position of equilibrium the work is still

$$W'' = 2ISZ.$$

The ratio of the works in the two latter cases,

$$\frac{W''}{W'} = \frac{Z}{H},$$

is equal to the tangent of the inclination. Hence, if we could measure these works, we could determine the elements of terrestrial magnetism without having recourse to magnets.

If the axis of rotation is in the magnetic meridian, the total work is null, when the current, which at first was in this plane, comes back to it after having been turned through 180°; the works, which correspond to the two halves of the rotation, are therefore equal and of contrary signs.

Lastly, the work would be null for any given rotation if the axis of rotation were parallel to the direction of the field.

489. Astatic Circuits.—The work of any given displacement is still null when the circuit comprises two closed curves, such that their projections on any plane give two equal surfaces S and S', surrounded by currents moving in opposite directions. This is the condition which movable currents must satisfy, which are arranged so as not to be under the action of the earth; what are called *astatic*

currents. Figures 115, 116, and 117 give examples of currents which realise these conditions.

If the two surfaces S and S′ were not equal, the action of the field would be proportional to their difference S – S′.

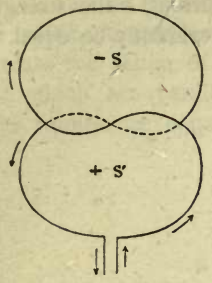

Fig. 115.

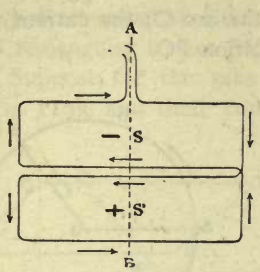

Fig. 116.

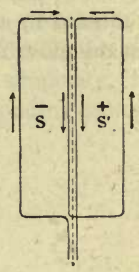

Fig. 117.

490. Rotation of a Current under the Action of the Earth.—A portion of a current not closed, and movable about an axis, takes in general a continuous rotatory motion under the influence of terrestrial magnetism.

We observe, in the first place, that in a uniform magnetic field, like the terrestrial field, we may always replace a current by its projection on three rectangular planes; this amounts, in fact, to replacing the strength of the field by its three rectangular components.

Consider any given current movable about an axis and determine its projections on three planes, one perpendicular to its axis of rotation, the two others passing through this axis, and such that one of them is parallel to the direction of the field; let S, S′, and S″ be these three projections. The projection S″ perpendicular to the field will not produce any action. The action on the projection S′ will be purely directive; the circuit of the current will be carried along in such a manner that the surface S″ is a maximum, and presents its negative face to the force; in our hemisphere the current must be descending in the part turned towards the east. There remains to be considered the projection S on the plane perpendicular to the axis. If it is closed, and of a fixed shape, it undergoes no action; if part of it is movable, the component of the field parallel to the axis will have a constant moment relatively to this axis, and will produce a continuous rotation.

491. For instance, let the system be formed of a current OP of length a (Fig. 118) movable about a vertical axis, one of whose ends O is on the axis of rotation and the other dips in a mercury cup. The current enters the mercury at A, traverses the two parts ABP and ACP in opposite directions, and regains the axis by the movable part PO. Let I be the total strength of the current, x the strength in the arc B, y in the arc C; the current will evidently be equal to I in the movable portion PO.

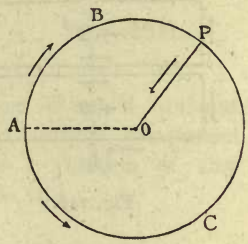

Fig. 118.

The surface comprised by the horizontal projection S consists of two parts, one ABPO presenting its negative face to the component Z of the terrestrial action; the other, ACPO, its positive face. The former tends to increase, the second to diminish, and for an angular displacement θ of the radius PO, the total work is

$$\frac{I}{2} a^2(x + y) Z\theta = \frac{I}{2} a^2 IZ\,\theta.$$

This work is independent of the position of the conductor OP; the force then is constant. The work corresponding to an entire turn will be

$$\frac{I}{2} a^2 2\pi IZ = SIZ.$$

If the current has a vertical projection S', the motion of rotation will be modified by the directive action corresponding to this projection. It is easy to see that according to the ratio of the two surfaces S and S', the initial velocity, and the value of friction, the moment of the directive action might preponderate over the moment of rotation,

and keep the apparatus in equilibrium in a position perpendicular to the magnetic meridian. In the apparatus used for this experiment, we take a movable current symmetrical in reference to the axis of rotation. The projection S′ is then null, and the couple of rotation, which would impart to the system a uniformly accelerated rotation if there were no friction, ultimately makes it rotate uniformly.

492. ACTION OF TWO RECTANGULAR CIRCUITS.—We may calculate the action of two rectangular frames AC and A′C′, the sides of which are parallel. Suppose, for the sake of simplicity, that the frames are equal (Fig. 119), and their corresponding summits

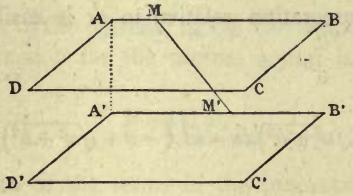

Fig. 119.

A and A′ on a perpendicular to their plane. The mutual energy of the two circuits, with currents equal to unity, is expressed on Neumann's formula (352) by

$$W = \int\int \frac{\cos\epsilon}{r} ds ds'.$$

The value of $\cos\epsilon$ is equal to unity for two parallel sides, and null for two perpendicular sides such as AB and B′C′. The energy thus becomes

$$W = \int\int \frac{ds ds'}{r} = \int ds \int \frac{ds'}{r}.$$

This expression only contains terms relating to parallel wires. Consider, in the first place, the two sides AB and A′B′, of length a, and at the distance h. Let ds and ds' be two elements, placed respectively at M and M′ and r their distance; lastly, suppose that we measure the lengths s and s' from the points A and A′. From the ratio

$$r^2 = h^2 + (s' - s)^2,$$

we get for the first integration, in which the distance s is taken constant,

$$\int \frac{ds'}{r} = \int_0^a \frac{ds}{\sqrt{(s'-s)^2+h^2}} = l.\left[s'-s+\sqrt{(s'-s)^2+h^2} \right]_0^a,$$

or

$$\int \frac{ds'}{r} = l.\frac{a-s+\sqrt{(a-s)^2+h^2}}{-s+\sqrt{s^2+h^2}}.$$

The second integration relative to ds is easily effected, for we have in general

$$\int l.\left(-u+\sqrt{u^2+h^2}\right)du = ul.\left(-u+\sqrt{u^2+h^2}\right)+\sqrt{u^2+h^2};$$

we get then

$$\int_0^a l.\frac{a-s+\sqrt{(a-s)^2+h^2}}{-s+\sqrt{s^2+h^2}}\,ds = 2\left[al.\frac{a+\sqrt{a^2+h^2}}{h} - \sqrt{a^2+h^2}+h \right].$$

Changing the sign of this expression and replacing h by the distance h' of the sides AB and C'D', we get in like manner the term relating to this last side. If the rectangle is a square, $h'=\sqrt{a^2+h^2}$, and the two terms of the energy corresponding to the side AB give

$$\left[al.\frac{a+\sqrt{a^2+h^2}}{a+\sqrt{2a^2+h^2}}\frac{\sqrt{a^2+h^2}}{h} + \sqrt{2a^2+h^2} - 2\sqrt{a^2+h^2}+h \right].$$

The total energy is then

$$W = 8\left[al.\frac{a+\sqrt{a^2+h^2}}{a+\sqrt{2a^2+h^2}}\frac{\sqrt{a^2+h^2}}{h} + \sqrt{2a^2+h^2} - 2\sqrt{a^2+h^2}+h \right].$$

When the distance of the frame is altered by dh, the variation of the energy dW, is equal to the work $-Fdh$ of the force F,

considered as attractive, which is exerted between the two circuits, and we have

$$F = -\frac{dW}{dh}.$$

We thus obtain, all reductions being made,

$$F = 8\left[\frac{a^2}{h\sqrt{a^2+h^2}} - \frac{a^2h}{(a^2+h^2)\sqrt{2a^2+h^2}} + \frac{2h}{\sqrt{a^2+h^2}} - \frac{h}{\sqrt{2a^2+h^2}} - 1\right].$$

If the strengths of the currents in the two frames are respectively I and I', the expression for the mutual action is

$$F = 8II'P,$$

where P is the sum of the terms in the brackets.

493. PROPERTIES OF CIRCULAR CURRENTS.—The potential of a circular current is equal, within a constant, to that of a shell of the same strength and the same contour. We have given above (368) the expression of this potential for any given point. If the point is on an axis at a distance x from the centre, it is sufficient in equation (16) to make $\rho = 0$; replacing Φ by I, we get

$$V = 2\pi I\left(1 - \frac{x}{u}\right) = 2\pi I\left(1 - \frac{x}{\sqrt{a^2+x^2}}\right);$$

from which is deduced

$$F = 2\pi I\frac{a^2}{u^3} = 2\frac{IS}{u^3},$$

denoting by S the surface of the circle. For a point on the axis the force is inversely as the cube of the distance to the contour.

This force is a maximum at the centre of the circle; we have then

$$F_m = 2\frac{IS}{a^3} = 2\pi\frac{I}{a} = \frac{IL}{a^2},$$

L being the length of the circumference.

This latter result would follow directly from a consideration of the equivalent shell. Let $2h$ be the thickness of the shell supposed to be plane, then denoting by I_a the intensity of magnetisation,

$$I = \Phi = 2hI_a.$$

The value of the action of the two terminal layers on a point at the centre is (322)

$$F = 4\pi I_a \left(1 - \frac{h}{a}\right),$$

and the magnetic induction is

$$F_1 = F_m = -F + 4\pi I_a = 4\pi I_a \frac{h}{a} = 2\pi \frac{I}{a}.$$

We may now reject the shell outside the point in question, without changing the value of the force (451).

494. Electromagnetic Solenoid.—Ampère gave the name *solenoid* to a system of equal circular currents, infinitely near, and infinitely close, equidistant and perpendicular to any given curve passing through their centre, which is called the *directrix*.

Let dS be the common surface of the elementary currents, h their distance, and I the strength of the current. Each elementary current may be replaced by a magnetic shell of the same magnitude, of thickness h, and surface density σ, such that we have

$$\Phi = h\sigma = I.$$

As the surfaces in contact of all these shells have equal and opposite charges, they neutralise each other except at the ends, and the system is identical with that of a solenoidal filament. The external action reduces then to that of two magnetic masses $\pm M$ placed at the ends. If l be the length of the solenoid, n the total number of elementary currents, and n_1 the number of these currents in unit length, we have

$$M = \sigma dS = \frac{I dS}{h} = \frac{n I dS}{l} = n_1 I dS.$$

495. Cylindrical Coil.—Let us suppose that a cylinder is covered with equidistant currents perpendicular to the axis. The system of these currents forms a kind of cylindrical solenoid, of

finite transversal dimensions; it is approximately realised by winding a wire in the form of a helix on the surface of the cylinder. Each element of the helix may be replaced by its projections on the axis and on a plane perpendicular to the axis. If the section of the cylinder is small, we sensibly destroy the effect of the former by bending the wire back in a contrary direction parallel to the axis.

Whatever be the diameter, if the individual turns are sufficiently near, and the coil consists of an equal number of layers in which the inclination of the windings is alternately in opposite directions, the effect of the projections on the axis is still sensibly zero, and the external action differs very little from that of the perpendicular projections.

The system of the currents perpendicular to the axis is equivalent to a solenoidal magnet of the same form; we may, in fact, replace each of them by a shell, and decompose the system into an infinity of parallel solenoids, each of which is equivalent to a solenoidal filament.

The action of the system on points outside the cylinder, reduces then to that of two equal and opposite layers spread uniformly on the bases, and the density σ of which is $n_1 I$.

For internal points the force is equal to the induction of the equivalent magnetic system. If the cylinder is so long that the action of the ends may be neglected in part of its extent, the lines of force are parallel to the axis of the cylinder; the field is uniform and its strength is

$$F_1 = 4\pi I_a = 4\pi n_1 I.$$

The flow of induction across the section of the cylinder is

$$Q = F_1 S = 4\pi n_1 I S;$$

this flow is in the opposite direction to the internal flow from the bases of the equivalent magnet.

It is, moreover, evident that a coil is not equivalent to a hollow magnet; in the hollow magnet all the lines of force, both internal and external, start from the positive surface, and are absorbed at the negative surface; in coils, on the contrary, the internal lines of force are the continuation of the external lines of force, and form closed curves which never terminate at magnetic masses.

496. ANNULAR COIL.—Suppose a ring to be covered by equal currents equidistant from each other, and each situate in a plane passing through the axis; the system may be decomposed into a series of solenoids, and it is equivalent to a solenoidal magnet of the same form (411).

All the elementary solenoids comprise then the same number of currents with the same intensity I, but of different lengths. If n_1 is the number of windings comprised between two meridian planes which make with each other an angle equal to unity, and if x is the radius of an elementary solenoid, the distance of the successive turns will be $\frac{x}{n_1}$; the intensity of magnetisation of the equivalent magnetic filament is then $\frac{n_1 I}{x}$, and the induction, or the magnetic force, $\frac{4\pi n_1 I}{x}$. The value of the flow of induction across a surface S, taken in the meridional section of the ring, is $4\pi n_1 I \int \frac{dS}{x}$.

In the case in which the ring is a circular torus (372), we have

$$\int \frac{dS}{x} = \pi(R - \sqrt{R^2 - a^2}).$$

The total flow across the section is then

$$Q = 4\pi^2 n I_1 (R - \sqrt{R^2 - a^2}).$$

497. Case of any given Surface.—Let us now consider the general case in which any surface Σ is covered by plane currents of the same strength, parallel to each other, and at such a distance that these are n_1 in unit length. These currents may be replaced by a series of parallel solenoids terminating in the surface, and these solenoids themselves by equivalent magnetic filaments; in this way a uniform magnet will be formed, the intensity of magnetisation of which is $n_1 I$, and the density at each point of the surface, has the value $n_1 I \cos\theta$, where θ is the angle which the perpendicular to the surface, at the point in question, makes with a perpendicular to the plane of the currents.

The internal action of these currents is equal to the induction of the equivalent magnetic system. In the case of the sphere (355) it is constant and equal to $\frac{8}{3}\pi n_1 I$; the value of the flow of induction across the great circle perpendicular to the line of the poles on the common axis of the currents is

$$Q = \frac{8}{3}\pi^2 R^2 n_1 I = \frac{2}{3} L^2 n_1 I,$$

where L is the circumference of the great circle.

The internal field would also be uniform in the case of an ellipsoid (356).

From this we arrive at a new way of regarding terrestrial magnetism; the magnetic action of the earth is equivalent to that of a series of circular currents situate in equidistant planes perpendicular to the magnetic axis, these currents circulating from east to west.

498. AMPÈRE'S THEORY OF MAGNETISM.—We see that it is possible, by means of currents situate in parallel planes, to realise a system equivalent to a uniform magnet, which has the same external surface; the two systems are equivalent for all external points, and produce the same induction in the interior.

Any given magnet may, in like manner, be replaced by a system of superficial currents, in so far, at least, as the external action is concerned.

This action, in fact (315), is equivalent to that of a layer of total mass null, distributed on the surface. If σ be the density of the layer at a point, F_n and F'_n the perpendicular components, measured from the surface, of the actions which it exerts outwards as well as inwards, we have (38)

$$4\pi\sigma = F_n + F'_n.$$

Let us consider the internal potential V′ of the layer, and the equipotential surfaces to which the force F′ is perpendicular, and suppose that on each of these surfaces we place equal and opposite magnetic layers, the density of which, at each point, is determined by the condition

$$\sigma' = \frac{1}{4\pi} F' = -\frac{1}{4\pi}\frac{dV'}{dn'}.$$

The external action of this system of surfaces is null. We observe now that the product $\sigma' dn' = -\frac{1}{4\pi} dV'$ is constant between two equipotential surfaces. If then we connect the negative layer of the surface Σ, where the potential is V′, with the positive layer of the following surface Σ', at the potential $V' + dV'$, we form a shell, the magnetic power of which, $\sigma' dn'$, is constant. A current, of the same strength, which followed on the surface, the curve formed by the intersection of the shell, would have the same action on the outside; we could proceed in the same way with all other shells. But in

forming this shell a negative corona has been left corresponding to the difference $\Sigma - \Sigma'$ of the two surfaces, and the sum of the external actions of these corona is equal and of opposite sign to that of the superficial currents. If $d\Sigma$ and dS are the two corresponding elements of this corona, and of the surface of the magnet determined by a tube of force, we have

$$F'_n d\mathrm{S} = F' d\Sigma = 4\pi\sigma' d\Sigma,$$

which gives

$$\sigma' d\Sigma = \frac{F'_n}{4\pi} d\mathrm{S}.$$

The quantity of magnetism contained in the element $d\Sigma$, is therefore the same as that which the element dS would contain with a density equal to $-\frac{F'_n}{4\pi}$.

It follows from this that the external action of all the surface currents defined by the preceding shells, is equivalent to that of a layer of total mass zero, the density of which at every point is $+\frac{F'_n}{4\pi}$.

By continuing the equipotential surfaces, which correspond to the external potential V, by continuous surfaces, we shall see in like manner that a layer whose density at each point is $+\frac{F_n}{4\pi}$, may be replaced by a system of surface currents.

The combination of the two systems of currents is a new system of surface currents having the same external action as a layer whose density is $\frac{1}{4\pi}(F_n + F'_n)$, that is to say, the density σ of the magnetic layer, which is equivalent to the proposed magnet.

The external action of any magnet may accordingly be replaced by that of a system of surface currents; but this equivalence does not hold for points in the interior.

It is possible, however, to obtain an exact representation of all the magnetic effects, both on the inside and on the outside, if we replace the magnetic filaments into which any given magnet may be decomposed (**313**), by the corresponding electromagnetic solenoids.

Ampère assumes that these elementary solenoids are formed of real currents circulating in the interior of molecules; these currents never pass from one molecule to another, and they exist before magnetisation, the only effect of which is to direct them.

On this hypothesis a magnet is no longer a continuous substance, but an assemblage of distinct molecules, giving rise to a very complicated distribution of the magnetic force and of the potential. But a material simplification results, for the force and the magnetic induction are then defined in the same way, and the force satisfies Laplace's equation, both inside and outside the magnet.

Ampère's hypothesis raises, however, a difficulty of prime importance, for it cannot be conceived that currents can permanently exist without the disengagement of heat, and therefore without a continued expenditure of energy. But, if the currents are supposed to be localised in the molecules themselves, the constitution of which is unknown, it is not impossible to assume that the resistance may be zero, and that the currents exist in a form which cannot be experimentally attacked. The assumption does not therefore necessarily imply a contradiction.

499. MAGNETISATION BY CURRENTS.—Magnetisation by currents was discovered by Arago in 1820. He had observed that a copper wire traversed by a current attracts iron filings; every particle of filing becoming a small magnet, places itself at right angles to the wire—the north pole on the left of the current, as in Œrsted's experiment. Ampère observed that the action of the current on a bar of soft iron or steel could be greatly increased by coiling the wire round the bar; in this way, and particularly with soft iron, temporary magnets are obtained which are called *electromagnets*.

Electromagnets may acquire a far greater power than that of even the best steel bars; but their principal characteristic is that of almost instantaneously gaining or losing their magnetic properties. They have further this curious property, that by suitably coiling the wire on the soft iron core, the magnetism may be distributed at pleasure, and any number of poles, or of *consequent points*, may be obtained on the bar.

The calculation of the effects of electromagnets is in general very difficult, even when the currents which surround the core are equidistant and parallel. In this case the system of currents develops an internal field, the strength of which is defined by the induction of a uniform magnet bounded by the same surface; the soft iron core placed in this field acquires a magnetisation at each point which depends on the strength of the field, and also on the magnetism developed by induction on the body itself. The magnetism can only be uniform then in the particular cases which we have examined (385). The external action is the resultant of that of the system of currents, or of the equivalent uniform magnet, and of that of the soft iron core.

500. EXAMPLES.—For a sphere surrounded by parallel and equidistant currents, the value of the strength of the field ϕ produced by the currents (497) is $\frac{8}{3}\pi n_1 I$.

The intensity of magnetisation is then (385)

$$\frac{3}{4\pi}\frac{\mu-1}{\mu+2}\phi = 2\frac{\mu-1}{\mu+2}n_1 I,$$

and the induction

$$F_1 = \frac{3\mu}{\mu+2}\phi = \frac{8\pi\mu}{\mu+2}n_1 I.$$

If the sphere is placed in a cylindrical coil which is so long that the effect of the ends may be neglected, it will also acquire a uniform magnetisation.

This would also be the case with an ellipsoid, and also with a sufficient approximation for a cylinder, the axis of which would coincide with that of the coil.

501. In the case of a cylindrical coil of great length (495), the strength of the field in the interior is $4\pi n_1 I$; the value of the intensity of magnetisation of a long cylinder parallel to the field would be $k\phi$ (292) or $4\pi k n_1 I$, and the internal induction F_1 will be $4\pi k\phi$, or $16\pi^2 k n_1 I$.

If S is the section of the bar, the flow of magnetic induction across it is

$$F_1 S = 16\pi^2 k n_1 IS,$$

and the total flow, including in it the induction $4\pi n_1 IS$ of the current, has the value

$$Q = 4\pi n_1 (1 + 4\pi k) IS.$$

This value can be experimentally investigated, and we might deduce from it the coefficient of magnetisation k.

502. The determination of this coefficient is still more accurate by means of a piece of soft iron in the form of a torus, which is surrounded by equidistant currents (496). In this case the strength of the field produced by the currents is $\frac{4\pi n_1 I}{x}$ at each

point. This being the only effective force, the value of the intensity of magnetisation at the point in question is

$$k\phi = \frac{4\pi n_1 \mathrm{I}}{x}.$$

The induction is equal to $4\pi k\phi$, so that the total flow of induction across the section S of the soft iron, comprising still that which arises from the currents, is

$$\mathrm{Q} = 4\pi n_1 \mathrm{I}(1 + 4\pi k) \int \frac{d\mathrm{S}}{x}.$$

If the soft iron does not occupy the whole of the space bounded by the currents, but merely a portion S′ of the section S, the total flow of induction, across the surface of the currents, is

$$\mathrm{Q}' = 4\pi n_1 \mathrm{I} \left[\int \frac{d\mathrm{S}}{x} + 4\pi k \int \frac{d\mathrm{S}'}{x} \right].$$

Suppose, for instance, that the section of the iron is a circle of radius a' concentric with the circular section of a torus; the total flow of induction in the torus will be

$$\mathrm{Q}' = 4\pi^2 n_1 \mathrm{I} \left[\mathrm{R} - \sqrt{\mathrm{R}^2 - a^2} + 4\pi k (\mathrm{R} - \sqrt{\mathrm{R}^2 - a'^2}) \right].$$

If the section of the coil were a rectangle of height b parallel to the axis of revolution, and of the thickness $2a$, the mean radius being R, we should have

$$\int \frac{d\mathrm{S}}{x} = b \int_{\mathrm{R}-a}^{\mathrm{R}+a} \frac{dx}{x} = bl. \frac{\mathrm{R}+a}{\mathrm{R}-a}.$$

The iron ring, in like manner, might have a rectangular section of height b', of thickness $2a'$, and of mean radius R′; the total flow of induction would then be

$$\mathrm{Q} = 4\pi n_1 \mathrm{I} \left[bl. \frac{\mathrm{R}+a}{\mathrm{R}-a} + 4\pi k b' l. \frac{\mathrm{R}'+a'}{\mathrm{R}'-a'} \right].$$

We shall see further on (559) the use which can be made of these various formulæ.

503. Measurement of Currents.—Galvanometers.—The strength of currents is usually measured by the electromagnetic or electrodynamic actions which they exert, and the instruments which are used for this purpose are called galvanometers, or electrodynamometers, according as they depend on one or the other of the two actions.

A galvanometer consists of a magnetised needle, or of any magnetic system on which a conductor traversed by a current is made to act; the effect produced is measured by means of an antagonistic force, such as the torsion of a metal wire, or of a bifilar suspension, or by the action of an external magnetic field.

Let us consider the simple case of a horizontal magnetic needle suspended by a wire without appreciable torsion, and placed in the centre of a frame on which is coiled a wire forming a series of parallel turns.

If the turns are parallel to the magnetic meridian, and they are traversed by a current, they produce a magnetic field, the strength of which is proportional to the strength of the current, and which may be represented by GI. The horizontal component of the terrestrial field at this point being H, the horizontal component of the field is $\sqrt{G^2I^2+H^2}$, and its direction makes an angle δ with the magnetic meridian, the tangent of which is equal to $\frac{GI}{H}$.

An infinitely small needle placed at this point, and which at first was in equilibrium in the plane of the needle, will be deflected through an angle δ, and from it we may deduce the strength of the current by the expression

$$I=\frac{H}{G}\tan\delta.$$

This formula is only exact provided the magnetic field is uniform throughout the whole space which the needle occupies. When the needle has a length which is considerable in reference to the dimensions of the frame, the intensity of the field is not constant, and the formula for the deflection is less simple. In that case, by an empirical graduation, we could determine the ratio which exists between the strength of the current and the deflection produced.

The magnetic moment of the needle has no influence on its position of equilibrium; it has no other effect than that of modifying the strength of the forces, and therefore the duration of the oscillations of the needle.

In order to increase the sensitiveness of the galvanometer—that is, the deflection δ for a given current—we must increase the value of G, and diminish that of H. The value of G is increased by increasing the number of turns by Schweigger's method, and by placing them as near the needle as possible. In order to diminish H, a magnet is placed at a certain distance, which produces at the centre of the frame a magnetic field parallel, and in the opposite direction to, that of the Earth.

Use is sometimes made of a quasi-astatic system of two needles (299), one of which is inside the frame and the other is outside; the action of the Earth on the movable system is then far feebler without there being any appreciable modification in the action of the current, which is exerted more particularly on the inner needle. We may also use two frames, each having one of its needles in the centre, and pass the current in opposite directions, so that the actions exerted on the two needles are concordant.

504. TANGENT GALVANOMETER.—In order to determine the absolute value of the strength of a current, besides knowing the component H of the terrestrial magnetism, we must also know the constant G of the galvanometer. The name *tangent galvanometer* is given to a galvanometer, the wire of which has been coiled in such a manner that this coefficient may be calculated from the dimensions of the wire and the shape of the frame.

If on the frame a wire L is coiled on a circle of radius a in such a manner as to make n turns, and if the needle, which is supposed to be infinitely small, is placed at a point of the axis at a distance u from the circumference, we shall have (493)

$$G = n\frac{2\pi a^2}{u^3} = \frac{La}{u^3} = \frac{4\pi^2 n^2}{L}\left(\frac{a}{u}\right)^3,$$

which gives

$$I = \frac{Hu^3}{La}\tan\delta = \frac{HL}{4\pi^2 n^2}\left(\frac{u}{a}\right)^3 \tan\delta.$$

The distance u is equal to a, when the needle is at the centre of the circle.

If the length is to be taken into account, we must estimate the strength of the field outside the axis of the currents by the formulæ of (368).

The formula of the tangent galvanometer would be exact and independent of the length of the needle if the field of the current were uniform. This would be the case, for instance, with a

cylindrical coil (495) or a spherical coil with equidistant currents (497). If n_1 is the number of turns for unit length, we have $G = 4\pi n_1$ in the first case, and in the second $G = \frac{8}{3}\pi n_1$.

505. Electrodynamometers.—In an electrodynamometer we measure directly the action exerted between two circuits, one fixed and the other movable, traversed by the same or by different currents. Suppose, for instance, that the magnet of a tangent galvanometer is replaced by a small coil, through which a current could be passed by a bifilar suspension, and which is in equilibrium when the axis of the coil is in the magnetic meridian. If a current I is passed through the wire on the frame of the galvanometer and a current I′ in the coil, the latter is displaced, and by a suitable torsion α of the suspension, it is restored to its original position.

The magnetic moment of the movable coil is proportional to I′, and may be represented by δi; the couple produced by the action of the frame is then GSII′. As the couple of torsion of the bifilar is proportional to the sine of the angle, if T is the moment of the couple which corresponds to an angle of torsion equal to $\frac{\pi}{2}$,

$$GSII' = T \sin \alpha.$$

If the two wires are traversed by the same current I, the expression becomes

$$I^2 = \frac{T \sin \alpha}{GS}.$$

Hence we might determine the strength of the current in absolute measure if we knew the constants T, S, and G, or we might leave these constants undetermined, and use the apparatus as an instrument of comparison. This is the principle of Weber's experiments.

If we suppose that the current traverses the parallel rectangular frames (492), as in Cazin's experiments, the intensity might be deduced from the attractive or repulsive action exerted between the two circuits.

506. Measurement of Discharges. — When the duration of the current is so short that the needle has no time to undergo an appreciable displacement before the current stops, it has nevertheless received an impulse or *throw*, and acquired a certain velocity; it is impelled from its position of equilibrium, and returns to it after a series of oscillations. This is the case,

for instance, of the discharge of a condenser through a conducting wire in which is a galvanometer; the total quantity of electricity may be deduced from the angle of throw imparted to the needle.

The strength of a permanent current in the galvanometer in question is given by an expression of the form

$$I = \frac{H}{G} f(\delta),$$

in which $f(\delta)$ reduces to the angle δ when the deflections are very small. If μ be the magnetic moment of the needle, the action of the current on the needle produces a couple, the moment of which is μGI.

We know, on the other hand, that when a body is movable about an axis, the product of the moment of inertia K, by the angular velocity is equal to the moment of the resultant couple in reference to the axis of rotation. Hence, since the deflection during the discharge is so small, that the action of the Earth can be neglected, we have for the needle in question,

$$K\frac{d\omega}{dt} = \mu GI.$$

If dm is the quantity of electricity which flows in unit time dt, this equation becomes

$$K\frac{d\omega}{dt} = \mu G\frac{dm}{dt}.$$

From which, if ω_0 is the initial angular velocity, and m the total discharge, we get

$$K\omega_0 = \mu Gm.$$

The needle, once impelled with this velocity ω_0, has a vis viva equal to $\frac{K\omega_0^2}{2}$, and it stops at an angle θ, when it has done work of the same value against the action of the terrestial field. We have then

$$\frac{K\omega_0^2}{2} = H\mu(1 - \cos\theta) = 2H\mu \sin^2\frac{\theta}{2},$$

or

$$\mu G^2 m^2 = 4HK \sin^2\frac{\theta}{2},$$

$$m = \frac{1}{G}\sqrt{\frac{HK}{\mu}}\, 2 \sin\frac{\theta}{2}.$$

If the deflections θ are small enough, we may simply take

$$m = \frac{1}{G}\sqrt{\frac{HK}{\mu}}\,\theta = \frac{H}{G}\,\theta\sqrt{\frac{K}{H\mu}}.$$

It appears thus that the angle of throw θ is proportional to the quantity of electricity which flows during the discharge, and this law of proportionality will be sufficient for all comparative experiments.

To determine m in absolute value we must know the constant G of the galvanometer, and the quantities which come under the root.

It may be observed that if the needle is left to itself under the influence of terrestrial magnetism, the time T of infinitely small oscillations is

$$T = \pi\sqrt{\frac{K}{H\mu}}.$$

from which follows

$$m = \frac{H}{g}\frac{T}{\pi}\,\theta.$$

As a matter of fact, the true angle of throw θ is diminished by the resistance of the medium, and by the induction currents which the motion of the needle produces in the wire. But if the oscillations do not diminish very rapidly, this effect is allowed for by adding to the angle θ, a quarter of the excess of this deflection over the deflection produced on the same side by the succeeding oscillation. We shall have, finally,

$$m = \frac{H}{G}\frac{T}{\pi}\left[\theta + \frac{1}{4}(\theta - \theta')\right].$$

CHAPTER IV.

INDUCTION.

507. FARADAY'S DISCOVERY.—The electromagnetic actions studied in the preceding chapters are purely *mechanical;* they are exerted on conductors traversed by currents, and correspond to a permanent condition of currents, and of the magnets near them. In all cases in which the systems experienced relative displacements, we have implicitly assumed that those displacements had no influence on the electric condition of the conductors. Faraday discovered in 1831 a class of phenomena of a totally different kind which corresponds to the variable condition of the system; these phenomena, which he comprised under the term *induction*, are of an *electrical* character, and are manifested by the production of temporary currents in conductors.

The currents which are formed are called *induced currents;* the *induced circuit* is that submitted to *induction;* the term inductor is applied to the current, the variation in which has been the cause of the induced current.

508. The phenomena discovered by Faraday may be classed under several heads:

1st. A closed circuit becomes the seat of a temporary current whenever a magnet is displaced near it; or if the magnetisation is varied; or still more generally when the magnetic field is modified in which the circuit is placed. This is *magnetoelectrical induction.*

2nd. Analogous effects are obtained by substituting a system of currents for the magnetic system. The circuit in question is traversed by an induced current whenever the distance, strength, or form of the external current is altered. The effect is the same as that which would produce the corresponding modification of the equivalent magnetic system or the current. This is *electrodynamic* or *voltaic induction.*

3rd. The change of form or of relative position of a closed

circuit, in reference to the magnetic field of a system of magnets or of currents, is ordinarily sufficient to give rise to an induced current in this circuit, which comes under one of the preceding heads.

4th. Finally, the mere fact of altering in any way the strength of the current in a circuit, even when it is withdrawn from any external action, produces an induction current in this circuit which adds itself to the principal current, and always tends to counteract the change of strength which it experiences; it is a current of self-induction or an *extra-current.*

509. Experiment has established the following general facts in reference to induction currents:

1st. Whatever be the kind of variation which gives rise to an induction current, two equal variations in opposite directions always give rise to equal and opposite currents.

2nd. The duration of the induced current is equal to that of the variation of the inducing system.

3rd. The quantity of electricity set in motion in the induced current by any operation is independent of the duration of the variation, and therefore of that of the induced current itself.

4th. Lastly, the nature of the conductor in which the induction currents are transmitted is only of importance in so far as it affects the resistance which it brings into the circuit.

510. Examining the various circumstances in which induction currents are produced, it is easily seen that their common characteristic is that of corresponding to a variation in the flow of magnetic force which traverses the induced circuit. This is evident for all the phenomena of the relative displacement of currents or of magnets; experiment shows, moreover, that any displacement, or any deformation of the induced circuit which does not modify the value of the flow which traverses it, never produces induced currents.

This is also the case with the extra current. For a current gives rise to a magnetic field, and therefore to a flow of force in the surface of the circuit which it traverses. It is easily seen that any change of intensity, or of shape, which modifies this flow, may produce an effect analogous to that which would be produced by the displacement of an external magnet, giving rise to the same variation.

We are thus led to define the phenomena of induction in the following manner:

When the flow of magnetic force which traverses a closed circuit is in any way modified, this circuit becomes the seat of a temporary current, the duration of which is equal to that of the variation of the flow.

This enunciation defines the conditions in which induced currents are produced. It remains for us to establish the direction and the magnitude.

511. LENZ'S LAW.—A short time after Faraday's discovery, Lenz enunciated the following law, which establishes a connection between the induction produced by the displacement of the inducing system, and the electromagnetic work as defined by Ampère's formula :

Any displacement of the relative positions of a closed circuit, and of a current or magnet, develops an induced current, the direction of which is such as would tend to oppose the motion.

512. NEUMANN'S THEOREM.—Lenz's law, which is of great practical utility, merely gives the direction of the induced current, but not the intensity. Assuming, as an experimental fact, *that the induction produced in a very short time is proportional to the velocity with which the conductor moves*, Neumann has given a complete theory of the induction currents produced in a movable linear conductor in the presence of any magnetic system. He has thus demonstrated this theorem, which we shall afterwards meet with under a more general form :

The electromotive force of induction is equal to the work which would be done in unit time by the magnetic system, if the intensity of the current in the induced circuit was equal to unity.

513. THEORY OF HELMHOLTZ AND THOMSON.—The existence of phenomena of induction may be considered as a necessary consequence of the conservation of energy combined with the electromagnetic law of Ampère and the law of Joule. This proposition was first put forth in 1847 by Prof. Helmholtz in his celebrated memoir on the *Conservation of force.* Sir W. Thomson arrived independently at the same conclusions.

Consider an invariable magnetic system, in the vicinity of a fixed conductor S, in communication with a battery. If the magnet is stationary, the strength I_0 of the permanent current is determined by Ohm's law, and if E is the electromotive force of the battery and R the resistance of the circuit,

$$E = I_0 R. \tag{1}$$

Multiplying both sides by $I_0 dt$, we get

$$EI_0 dt = I_0^2 R dt. \tag{2}$$

This equation expresses that during the time dt, the energy due to the chemical actions is equal to the thermal energy expended in the circuit on Joule's law.

Suppose now that instead of being stationary, the magnetic system moves in accordance with electromagnetic actions. The external work resulting from this displacement, can only be borrowed from the sole source of energy in the system (that is, the chemical action), and the preceding equation must be in default. On the other hand, there is no reason for supposing that the laws of Faraday and Joule cease to hold; in other words, the weights of the bodies combined in the different couples must still be proportional to the strength of the current, and the thermal energy disengaged in the circuit is equal to the product of the resistance into the square of the strength. Hence, the strength of the current could not retain its original value.

514. Suppose now that the magnet is displaced in such a way that the new value of the strength remains constant. So long as this condition is fulfilled, the excess of chemical work over the thermal energy expended in the circuit in the time dt, serves to produce the external work $d\mathrm{T}$ corresponding to the electromagnetic forces. Hence, if I is the strength of the current,

$$\mathrm{EI}dt = \mathrm{I}^2\mathrm{R}dt + d\mathrm{T}. \tag{3}$$

If Q is the flow of force due to the magnetic system which traverses the circuit, entering by its negative face, we have

$$d\mathrm{T} = \mathrm{I}d\mathrm{Q}.$$

Replacing $d\mathrm{T}$ by this value in the equation (3), and dividing by $\mathrm{I}dt$, we get

$$\mathrm{E} = \mathrm{IR} + \frac{d\mathrm{Q}}{dt}. \tag{4}$$

In order that the strength of the current shall be constant, the displacement must take place so that the differential $\frac{d\mathrm{Q}}{dt}$ is itself constant.

Putting $\mathrm{I}_0 - \mathrm{I} = i$, equations (1) and (4) give

$$(\mathrm{I}_0 - \mathrm{I})\mathrm{R} = i\mathrm{R} = \frac{d\mathrm{Q}}{dt}. \tag{5}$$

It will be seen that the differential $\frac{dQ}{dt}$ plays the part of an electromotive force acting in the contrary direction to E, and capable of producing a current i in the contrary direction to the principle current, so that the resultant current I still satisfies Ohm's law, under the form

$$E - \frac{dQ}{dt} = IR. \tag{6}$$

The quantity $\frac{dQ}{dt}$ is called *the electromotive force of induction; it is equal to the differential, in respect of time, of the flow of magnetic force which traverses the circuit.*

If the value of dQ is positive—that is to say, if the flow of force increases, the electromotive force of induction diminishes the strength of the current, and the work of the electromagnetic forces is positive. If, on the contrary, the value of dQ is negative, the magnet is displaced in resisting the electromagnetic forces, and this operation introduces a fresh energy into the system; the strength of the current is then greater than in a state of rest.

515. The quantity dm or idt of electricity induced in the wire is given by the equation

$$iRdt = Rdm = dQ. \tag{7}$$

The total quantity of electricity m corresponding to a finite displacement, for which the flow of force passes from the value Q_1 to the value Q_2, is therefore

$$m = \frac{Q_2 - Q_1}{R}. \tag{8}$$

516. The establishment of the current in a circuit requires itself work which we have not taken into account, and this work (to which we shall subsequently revert) is a function Ψ of the strength of the current. During the variable period, the energy of the chemical action should also furnish the work $d\Psi$ which corresponds to an increase dI of strength. Equations (3) and (5) then become

$$\begin{aligned} EIdt &= I^2Rdt + IdQ + d\Psi, \\ iR &= \frac{dQ}{dt} + \frac{1}{I}\frac{d\Psi}{dt}. \end{aligned} \tag{9}$$

We get from this, by a reasoning analogous to that which gave equation (8),

$$(8)' \qquad mR = Q_2 - Q_1 + \int_{\Psi_1}^{\Psi_2} \frac{I}{\bar{I}}\, d\Psi.$$

Whatever be the law by which the magnetic system is displaced, if the strength of the currents is the same at the two limits, the last term of equation (8′) is null. This is the case more particularly if the limits are chosen before and after the motion, in which case the two limiting values of the current are equal to I_0.

With these limitations we may enunciate in a general form the theorem expressed by equation (8):

The total quantity of electricity put in motion by any displacement of a magnetic system, is equal to the quotient of the variation of the flow of force corresponding to this displacement, by the resistance of the circuit.

517. The preceding results suggest some important remarks.

1st. It is seen, in the first place, that the electromotive force of induction is of a kind which opposes the motion, for the original intensity of the current is diminished or increased according as the magnetic system obeys or resists electromagnetic actions; this is *Lenz's law.*

2nd. If the strength of the current were equal to unity, the external work $d\mathrm{T}$ would correspond to the work $\frac{dQ}{dt}$ in unit time. The electromotive force of induction is equal to this work; this is *Neumann's theorem.*

3rd. The electromotive force of induction is independent of the electromotive force E of the battery; the induction is then the same however feeble is the strength of the original current. It results from this, that induction should also take place when the conductor is neutral, provided it forms a closed circuit. It is in this form that Faraday discovered *induction currents.*

It must, however, be observed that if the current was really null throughout the entire extent of the conductor, the reciprocal action of the magnet and of the circuit would also be null, and the preceding considerations would not enable us to foresee the production of induced currents. But it may be said that this perfect neutrality is a state of unstable equilibrium, impossible to realise in practice, and that an infinitely slight cause, a change of temperature at any point of the circuit, or the displacement of an external electrified body, even at a great distance, would be sufficient to produce a current, however slight, in the conductor, and thus enable induction to take place.

518. General Law of Induction.—The preceding reasoning would apply in the same way, and in almost identical terms, to the other cases of induction.

It is seen to be evident for electromagnetic induction—that is to say, that which is produced by the displacement of a system of constant currents, substituted for the magnetic system—for we have demonstrated the complete equivalence of the magnetic fields produced by currents and by magnets.

In the case of currents induced by a variation in the strength of a magnet, or of an adjacent current, the result may be considered as equivalent to that which would be obtained by bringing a magnet or a current, identical with the variation in question, from an infinite distance to superpose it on the former.

Experiment shows that extra-currents, produced by deformations of the circuit itself, or by changes in the strength of the principal current, are also connected, and in the same manner, with the corresponding variations of the flow of magnetic force.

It may therefore be considered as a general rule that any variation in the flow of force in a circuit, whatever be its origin, corresponds to a variation of potential energy, and gives rise to the same electromotive force of induction as if it were produced by the displacement of an external magnetic system.

This conclusion appears necessary if, abandoning the idea of actions at a distance, we regard the transmission of electrical and magnetic forces as due to a modification of the elastic properties of the medium; we can understand then that the only proximate cause of currents induced in a conductor may be the state of the medium in which is the conductor, whatever may be the origin of the forces which are at work in this medium. We may, therefore, formulate the general law of induction phenomena in the following terms:

The total electromotive force developed in a circuit at a given time is equal to the differential, in regard to time, of the flow of magnetic force across it.

Or again: *The total quantity of electricity induced in a circuit, is equal to the product of the inverse of its resistance by the total variation of the flow of force across it.*

The flow of force across a circuit at a given time consists of the flow Q, arising from external bodies, magnets or currents, and of the flow produced by the current which traverses the circuit itself. Let L be the value of this latter flow when the intensity of the current is equal to unity; it will be equal to LI for strength I, and if E are

the ordinary electromotive forces at work in the circuit, the general equation of induction will be

$$(10) \qquad (E - RI)dt = d(Q + LI),$$

or

$$E - \frac{d(Q + LI)}{dt} = RI.$$

519. Coefficients of Induction.—If the inducing system is a magnetic shell or a current, the flow Q is equal to the product of a constant M, by the magnetic power of the shell or the strength of the current. This constant is a function merely of the form and relative position of the two circuits; we know that it has the same value for the two adjacent conductors (341), and that its value is defined (353) by the integral

$$(11) \qquad M = -\int\int \frac{\cos\epsilon}{r} ds ds'.$$

This factor M is called the *coefficient of reciprocal induction*, or *of mutual induction* of the two circuits.

The constant L is an integral of the same form but with this difference, that the two elements ds and ds' belong to the same circuit. It is called the *coefficient of self-induction.*

The value of the coefficient of self-induction L is the limit towards which M tends when two equal circuits traversed by currents in the same direction, and of the same strength I, nearly coincide. For the total flow, which at this instant traverses the system of the two circuits, is equal to the sum of the flows produced by each of them—that is to say, to 2LI; it may also be considered as the sum 2MI of the equal flows, each one of which starts from one of the circuits and traverses the positive face of the other.

520. Electromagnetic Induction.—The general formula, applied to the case in which the inducing system is a shell of power Φ, gives

$$(12) \qquad (E - RI)dt = d(M\Phi + LI).$$

Suppose that while the magnetic power Φ is constant, the shell is brought from an infinite distance to a determinate position in presence of the induced circuit which we suppose fixed; we have then

$$(E - RI)dt = \Phi dM + L dI.$$

If both sides of this equation are multiplied by I, and we integrate from $t=0$ to the time t when the shell takes up its final position, we get

$$(13) \qquad \int_0^t (EI - RI^2)dt = \Phi \int_0^t I\frac{dM}{dt}dt + \left(L\frac{I^2}{2}\right)_0^t.$$

The first member of this equation represents the excess of the chemical energy, furnished by the battery during the time t, over the energy which appears as heat in the circuit during the same time.

The first term of the second member is the total work of the electromagnetic actions; this work depends on the law of the motion. We may imagine, for instance, that the shell may have been approached very slowly, so that the induction is very feeble, and the principal current differs very little from its initial value I_0; in this case, if M is the flow of force corresponding to the final position of the shell, the electromagnetic work will be equal to $\Phi M I_0$.

This latter term represents the change in the *potential energy* of the current; it is zero if the two limiting values of the current are the same—that is, if the magnetic shell is in a state of rest in its final position.

521. If the power of the magnetic shell, while still at rest, was variable, the equation would be quite analogous:

$$(E - RI)dt = Md\Phi + LdI,$$

$$\int_0^t (EI - RI^2)dt = M\int_0^t I\frac{d\Phi}{dt}dt + \left(L\frac{I^2}{2}\right),$$

and would lead to the same conclusions.

522. The most general case is that in which the three functions M, Φ and L vary simultaneously—that is to say, when the magnetic shell changes its strength, its form, and relative position, and that the circuit itself is deformed. Equation (12) gives therefore the electromotive force of induction

$$(14) \qquad e = \frac{d(M\Phi + LI)}{dt} = M\frac{d\Phi}{dt} + \Phi\frac{dM}{dt} + L\frac{dI}{dt} + I\frac{dL}{dt}.$$

If the induced circuit contains no electromotive force independent of induction, we need only make $E=0$ in the preceding equations.

523. Electrodynamic Induction.—If the inducing system were a constant current, we might replace it by the equivalent magnetic shell, and thus bring it within the preceding case; but in consequence of reactions, the inducing circuit itself will be under induction, and the strength of the current will no longer be constant. If R′ and L′ are the resistance and the coefficient of self-induction of the inducing circuit, and E′ the electromotive force which it contains, the strength of the current in the two circuits will be determined at each instant by two simultaneous equations

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$$(E - RI)\,dt = d(MI' + LI),$$
$$(E' - R'I')\,dt = d(MI + L'I'). \tag{15}$$

The complete solution of these equations generally presents great difficulties, and in the next chapter we shall investigate the simplest cases in which it can be obtained; but the differential equations already suggest some important remarks.

If we add these equations, after having multiplied the first by I and the second by I′, we get

$$(EI + E'I' - RI^2 - R'I'^2)\,dt = I\,d(MI' + LI) + I'\,d(MI + L'I'). \tag{16}$$

The left hand side represents the excess of the energy furnished by the sources in the two circuits over the thermal energy expended in the conductor.

The right hand side may be written as follows:—

$$\frac{I}{2}\,d(LI^2 + 2MII' + L'I'^2) + \frac{I}{2}\,I^2 dL + II'dM + \frac{I}{2}\,I'^2 dL'; \tag{17}$$

it represents the total variation of the potential energy of the two circuits, and the external work.

524. Intrinsic Energy of the Current.—If the circuits are fixed both in form and position, the factors L, M, and L′ are constants; the portion of the energy not converted into heat is expressed by

$$d\left[\frac{LI^2}{2} + MII' + \frac{L'I'^2}{2}\right].$$

The term MII′ is the relative energy of the two currents; it is the work which would have been necessary to bring the circuits traversed by the currents I and I′ from an infinite distance to their actual position. Each of the other terms within the parenthesis may be called the *intrinsic energy* of the corresponding current; it is equal to half the product of the coefficient of self-induction by the square of the strength, and represents the cost of the work of creating the actual current in each circuit (this not being subject to any foreign action), or the external work which this current could develop if it were left to itself, and vanished under the same conditions.

It may be observed that we may write

$$\frac{LI^2}{2}+MII'+\frac{L'I'^2}{2}=\frac{1}{2}(LI+MI')I+\frac{1}{2}(L'I'+MI)I'.$$

Each of the two terms on the right hand side represents the potential energy of the corresponding circuit; it is the work which must be spent in each circuit when the field is brought from zero to its actual state, and is therefore the work which it would produce if all the currents were simultaneously annulled. In each of the circuits this work is equal to half the product of the strength by the flow of magnetic force which traverses it.

That part of the energy which we have been here considering is in a form which it is not possible to define in the present state of science. We cannot say, for instance, whether it is in the state of ordinary potential energy, like the tension of an elastic body, or of an actual energy consisting in the motion of a particular fluid, or again in both of these at once; nor further, whether it is localised in the circuit traversed by the current, or diffused through the whole medium, in accordance with the ideas of Faraday and of Maxwell.

525.—In the general case in which the factors L, M and L′ are variable, the first term of the expression (17) always represents the variation in the potential energy of the two circuits; the whole of the other terms

$$\frac{1}{2}I^2dL+II'dM+\frac{1}{2}I'^2dL'$$

represents the work done by the electrodynamic actions of the conductors, in consequence of their changes of form, or of relative position.

Suppose that the two circuits have an invariable shape, and that they are so displaced that the two strengths I and I′ retain constant values, which are naturally different from the initial or final values; as the coefficient M is the only one which changes, the term for the potential energy and that of the external work are both reduced to the same value $II'dM$. We have then, at each moment,

$$(EI + E'I' - RI^2 - R'I'^2)dt = 2II'dM.$$

It will thus be seen that the excess of the chemical energy furnished by the pile over the energy expended in heat, is equal to twice the external work expended in effecting the displacement; half this energy is used in producing external work, the other in increasing the potential energy of the system.

This remark, which is due to Sir W. Thomson, should be compared with the analogous proposition relative to the displacement of conductors at constant potentials.

CHAPTER V.

PARTICULAR CASES OF INDUCTION.

526. ELECTROMAGNETIC RESISTANCE IS A VELOCITY.—Consider the case (Fig. 114) of a bar CC′ sliding parallel to itself on two parallel rails AA′, BB′, at the distance b, situate in a vertical plane at right angles to the magnetic meridian, the ends of which are connected by a metal conductor. Suppose that the direction of the horizontal component of the terrestrial field is from back to front. When the bridge CC′ is moved away from AB, parallel to itself, it is carried in the direction in which the action of the Earth would urge it, if the circuit were traversed by a current going from A to B by the bridge. This motion produces a current of induction, which traverses the circuit in the opposite direction—that is, which goes from B to A by the bridge.

If we consider that the resistance of the rails may be neglected in comparison with that of the conductor which joins the points A and B, and if R is the resistance of the circuit which we suppose to be unchanged, x_1 and x_2 the values of the distance AC in the two successive positions of the bridge, and H the horizontal component of the terrestrial magnetic field, the corresponding quantity M of induced electricity will be given by the equation

$$RM = Q_1 - Q_2 = bH(x_1 - x_2).$$

As the product $b(x_1 - x_2)$ represents the area S described by the bridge, it follows that

$$R = \frac{SH}{M}.$$

In this expression the factor H is the intensity of a magnetic field—that is to say, a force which is exerted upon unit mass;

hence, if r and l are two lengths, and m a magnetic mass,

$$SH = \frac{m}{r^2} l^2.$$

On the other hand, the electrical mass M is the product by a time of an intensity of current, or of the magnetic power of a shell, and we may also write

$$M = It = \Phi t = h\sigma t = h \frac{m'}{l'^2} t,$$

m' denoting a magnetic mass of suitable magnitude, and h and l' lengths. We shall have then

$$R = \frac{m}{m'} \frac{l^2}{r^2} \frac{l'}{h} \frac{l'}{t}.$$

As the three first fractions are abstract numbers, it will be seen that the resistance expressed in electromagnetic units is equal to the quotient of a length by a time—that is to say, a quantity of the same order as a velocity.

We may, indeed, easily discover in the experiment itself, a physical representation of this velocity. Suppose, in fact, that the cross-bar moved uniformly with a velocity u, and that the intensity I of the current is measured by the action which it exerts on a needle placed in the centre of a tangent galvanometer (504); we shall have

$$\tan \delta = \frac{L}{a^2} \frac{I}{H}.$$

On the other hand, $MR = HS$, or

$$RIt = Hbut;$$

it follows from this that

$$\tan \delta = \frac{L}{a^2} \frac{bu}{R}.$$

If the velocity u is so great that the action of the current is equal to that of the Earth—that is, that the deflection of the needle in the galvanometer is 45°—we shall have

$$R = \frac{Lb}{a^2} u,$$

and if we take $a^2 = Lb$, $R = u$.

Hence the resistance of the circuit in question is equal to the velocity with which the bridge must be uniformly moved under the given conditions, in order that the action of the current induced in a galvanometer of suitable dimensions may produce a deflection of 45°.

527. The following experiment, suggested by Faraday, may be considered as an application of the same problem.

Suppose that two electrodes, A and B, are immersed in water on the opposite edges of a river, of a canal, or of a current in the sea, and are connected by a metal conductor. If u is the velocity of the current, and a the distance of the electrodes, under the influence of the Earth's magnetism, an electromotive force equal to uZa will be established between them, which, in a circuit of resistance R, will develop a current of intensity $\frac{uaZ}{R}$. The experiment is not impracticable; but, unless we could work with very great values of u and a, the polarization of the electrodes would no doubt make it very difficult to verify the conclusion.

528. Closed Circuit in a Uniform Field. — Consider a closed circuit, which may be supposed plane (487), and let S be its surface. Let us suppose it placed in a uniform field of intensity F—the terrestrial field, for instance—and perpendicular to the direction of the field. If the frame be made to turn through an angle α, the variation of the flow of magnetic force is equal to $FS(1 - \cos\alpha)$, and the quantity M of electricity put in motion in the circuit, which we suppose has the resistance R, is given by the ratio

$$M = \frac{FS(1 - \cos\alpha)}{R}.$$

If the frame turns through 180°, face for face, we have

$$M = \frac{2FS}{R}.$$

This quantity of electricity may be measured in absolute value (506) by the throw of the needle of the galvanometer, which would enable us to determine the intensity of the magnetic field.

The direction of the current on Lenz's law, is that which ought to traverse the current in its original position, in order that it may be in stable equilibrium.

529. Determination of the Inclination by Induction Currents.—We have seen (487) that the tangent of the magnetic inclination is equal to the ratio of the works which for the same strength of current correspond to a rotation of 180° of the circuit, starting from a position at right angles to the magnetic meridian: 1st, about a horizontal axis perpendicular to the meridian; 2nd, about a vertical axis. This ratio is that of the electromotive forces of induction for the same displacements, and therefore that of the corresponding quantities of induced electricity (506); a measurement of this latter ratio will therefore give the inclination.

530. Faraday's Disc.—A metal disc, movable in a uniform field about an axis parallel to the direction of the field, forms part of a circuit which communicates on the one hand with the axis of rotation, and on the other with a spring which presses on a point of the circumference. When the disc is put in uniform rotation, a uniform current is also produced in the circuit. It will be seen that the arrangement of this experiment is, as it were, the inverse of that of Barlow (483). If the plane of the disc is vertical, and the force of the field F traverses it from front to back, and if the direction of the rotation is that of the hands of a watch, the induced current traverses the disc from the centre to the edge.

If a is the radius of the disc, and ω the angular velocity, the electromotive force is

$$e = \frac{dQ}{dt} = \frac{\omega a^2}{2} \mathrm{F}.$$

An analogous result is obtained, though with a less simple calculation, by placing the disc between the poles of a horse-shoe magnet or between the armatures of two electromagnets. In this latter form, M. Le Roux obtained currents so strong that bright sparks passed between the disc and the spring.

Moreover, all the experiments, particularly those which were examined in Chap. III., in which the motion of a conductor is produced by electromagnetic or electrodynamic actions, would produce an inverse induction current if the motion was kept up by an extraneous cause.

531. TERRESTRIAL CURRENTS.—Let us consider, for example, a sphere magnetised uniformly. Let us suppose that a conducting arc, resting with one end at the pole and the other on the equator, turns about the axis with a uniform motion; this arc will cut the same flow of force as if it were applied on the surface along a meridian. An element ds, the velocity of which is v, cuts in each unit of time, a flow of force equal to $Zvds$, Z being the perpendicular component of the magnetic force on the corresponding parallel. If V be the velocity at the equator, F_p the magnetic force at the pole, a the radius of the sphere, and λ the latitude of the element ds, we have

$$v = V \cos \lambda, \qquad Z = F_p \sin \lambda, \qquad ds = a d\lambda.$$

The flow of force cut in each unit of time by the whole arc is equal to the electromotive force of induction e, which gives

$$e = \int_0^{\frac{\pi}{2}} VF_p a \sin \lambda \cos \lambda \, d\lambda = \frac{1}{2} F_p V a.$$

If the arc is insulated, this value of e represents the difference of potentials at the two ends.

If the ends of the arc were connected to two bodies of the same capacity C, these bodies would acquire, after a longer or shorter time, equal and opposite statical charges, the absolute value of which would be $\frac{1}{2}Ce$.

Finally, if the arc were closed by a fixed conductor on the inside or outside of the sphere, the circuit would be traversed by a continuous uniform current, from the equator to the pole or conversely.

As the Earth may be compared to a sphere magnetised uniformly, it will be seen that an external arc, which does not share the rotatory motion, should be traversed by an induction current from the equator to the pole, for the direction and magnitude of the induced currents only depend on the relative motion of the arc and of the magnetic system. It is probable that this induction plays an important part in certain natural phenomena, such as the aurora borealis (which seem to be electrical discharges in the upper regions of the atmosphere), the currents observed on the surface of the Earth in telegraphic wires, and the perturbations of the magnetic elements.

532. VARIABLE STATE OF A CURRENT.—The establishment of a current in a circuit represents a certain amount of work which is the potential energy of the current; this energy is absorbed at the starting of the current, and is restored when the electromotive forces disappear. In all cases, the effects of self-induction, which are the consequence, determine the law of intensity during the variable period, whether at the closing or opening of the circuit.

Consider a single circuit. Let R be the total resistance, L its coefficient of self-induction, and E the electromotive force which it contains. We have the equation

$$E = \frac{d(LI)}{dt} + RI. \tag{1}$$

If we suppose L constant as well as E, the strength at each moment is given by the formula

$$I = I_1 + (I_0 - I_1)e^{-\frac{Rt}{L}}, \tag{2}$$

I_0 being the initial value and I_1 that which corresponds to the permanent state.

The total quantity of electricity which passes in time t, is

$$\int_0^t I\,dt = I_1 t + (I_0 - I_1)\frac{L}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

If the time t is sufficiently great, we have simply

$$\int_0^t I\,dt = I_1 t + (I_0 - I_1)\frac{L}{R}. \tag{3}$$

We have also, for a sufficiently long time,

$$\int_0^t I^2\,dt = I_1^2 t + (I_0 - I_1)\left(\frac{I_0 + 3I_1}{2}\right)\frac{L}{R}; \tag{4}$$

this expression is proportional to the calorific energy expended in the circuit.

It may be observed that these values of the two integrals are the same as if there had been a current of strength $\frac{I_0+I_1}{2}$ during the time $\frac{2L}{R}$, which had been succeeded by a current of the normal intensity I_1, during the rest of the time.

Suppose that the electromotive force is constant, and that we measure the time from the closing of the circuit; we have then

$$I_0=0, \qquad I_1=\frac{E}{R},$$

and therefore

$$I=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right). \tag{5}$$

533. The expression $\frac{E}{R}e^{-\frac{Rt}{L}}$ represents the strength of the extra current obtained after the moment t. It will be seen that the current only attains its normal strength after an infinitely long time; but if the ratio $\frac{R}{L}$ is very great, which it is in most cases, the exponential tends rapidly towards zero, and after a very short time, the real strength only differs from the final strength, by a quantity which may be neglected. In order to calculate after what time this difference would be below a given quantity, $\frac{1}{n}$ for instance, we may put

$$e^{-\frac{Rt}{L}}=\frac{1}{n},$$

from which we deduce

$$t=\frac{L}{R}l.n.$$

The total quantity of electricity which corresponds to the extra current is

$$\frac{E}{R}\int_0^\infty e^{-\frac{Rt}{L}}dt=\frac{EL}{R^2}; \tag{6}$$

it is the same as if the current had had half the intensity of the normal value $-\frac{1}{2}\frac{E}{R}$ in the time $\frac{2L}{R}$.

534. Extra Current on Opening.—Suppose that the permanent regime being established, we suddenly introduce a resistance r in the circuit; we shall have at the two limits

$$I_0 = \frac{E}{R},$$

$$I_1 = \frac{E}{R+r},$$

and the strength at any instant is given by the equation

$$(7) \qquad I = \frac{E}{R+r}\left(1 + \frac{r}{R}e^{-\frac{(R+r)t}{L}}\right).$$

The value of the total quantity of electricity which corresponds to the extra current is

$$(8) \qquad \frac{Er}{(R+r)R}\int_0^\infty e^{-\frac{(R+r)t}{L}}dt = \frac{EL}{(R+r)^2}\frac{r}{R}.$$

This case has some analogy with that in which the circuit is broken in air, the resistance r being that of the layer of gas traversed by the spark on breaking; but in reality this resistance is far from being constant while the phenomenon lasts.

Suppose that instead of breaking the circuit we had separated it from the battery, by replacing the latter by a wire of the same resistance, so that the total resistance of the circuit is still represented by R, which simply amounts to suppressing the electromotive force. Equation (1) reduces to

$$(9) \qquad L\frac{dI}{dt} + RI = 0.$$

Determining the constant by the conditions that for $t=0$ we have $I = \frac{E}{R}$, it follows that

$$(10) \qquad I = \frac{E}{R}e^{-\frac{Rt}{L}}.$$

In this case the law of the extra current of opening is the same as that of the current of closing (533), and the quantities of electricity put in motion are the same.

535. VARIABLE ELECTROMOTIVE FORCE. — Suppose that the electromotive force, instead of being constant as with ordinary batteries, undergoes periodical variations, and is represented by an expression of the form

$$E = E_0 \sin 2\pi \frac{t}{T}. \tag{11}$$

When the steady condition is obtained, the intensity of the current evidently follows the same period, and may be represented by the expression

$$I = A \sin 2\pi \left(\frac{t}{T} - \phi \right). \tag{12}$$

Substituting this value in equation (1), and determining the constants A and ϕ, by the condition that it shall be satisfied for any given time, we find

$$A^2 = \frac{E_0^2}{R^2 + \frac{4\pi^2 L^2}{T^2}}, \tag{13}$$

and

$$\tan 2\pi\phi = \frac{2\pi L}{T R}. \tag{14}$$

It will be seen from this that the effect of the coefficient of self-induction is to increase the apparent resistance of the circuit. The strength of the current is zero whenever $t - \phi T = n\frac{T}{2}$. The value ϕT expresses the time which elapses from the time at which the electromotive force is zero, and that when the current itself passes zero. It is a kind of *retardation* to the transmission of the electromotive force, which arises solely from the effects of induction.

Whatever be the values of L, of R, and T, the maximum value of $2\pi\phi$ is $\frac{\pi}{2}$, that is to say, that the difference of phase ϕ is equal to $\frac{1}{4}$. The maximum retardation of the induced current is therefore equal to a quarter of the whole period, or to half the semi-period.

During a semi-period, the quantity of electricity which passes through the circuit is

$$(15) \qquad Q = A \int_0^{\frac{T}{2}} \sin 2\pi \frac{t}{T} dt = \frac{AT}{\pi},$$

and the corresponding calorific work

$$(16) \qquad W = RA^2 \int_0^{\frac{T}{2}} \sin^2 2\pi \frac{t}{T} dt = R \frac{A^2 T}{4}.$$

From this is deduced, for the mean intensity I′ of the current, disregarding the sign,

$$(17) \qquad I' = \frac{2A}{\pi},$$

and for the mean strength I″, which would give the same quantity of heat

$$(18) \qquad I'' = \frac{A}{\sqrt{2}}.$$

536. CURRENT OF DISCHARGE—OSCILLATING DISCHARGES.—Let us finally consider, with Sir W. Thomson, the phenomena which accompany the discharge of a conductor. Let C be the capacity of the electrified body in electromagnetic measure, Q_0 its charge; it is connected with the ground by a wire of resistance R, and the coefficient of self-induction of which is L. At a given moment t, the charge of the conductor is Q, and its potential $\frac{Q}{C}$, which gives the equation

$$\frac{Q}{C} = L \frac{dI}{dt} + RI.$$

Observing that $I = -\frac{dQ}{dt}$, this may be written in the form

$$(19) \qquad \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{CL} Q = 0.$$

The general integral of this equation is

$$Q = Ae^{\rho t} + A'e^{\rho' t}, \tag{20}$$

ρ and ρ' being the roots of the quadratic equation,

$$\rho^2 + \frac{R}{L}\rho + \frac{1}{CL} = 0, \tag{21}$$

which gives

$$\rho = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}.$$

According as $L \lesseqgtr \frac{R^2C}{4}$, these roots are real or imaginary.

The constants A and A′ are determined by the condition that for $t = 0$, we have $Q = Q_0$, and $I = 0$, which gives $0 = A\rho + A'\rho'$.

If the roots of the equation (21) are real, then representing by α the radical $\sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$,

$$Q = Q_0 e^{-\frac{Rt}{2L}} \left[\left(\frac{1}{2} + \frac{R}{4\alpha L} \right) e^{\alpha t} + \left(\frac{1}{2} - \frac{R}{4\alpha L} \right) e^{-\alpha t} \right],$$

$$I = \frac{Q_0}{2\alpha LC} e^{-\frac{Rt}{2L}} \left(e^{\alpha t} - e^{-\alpha t} \right). \tag{22}$$

When the roots are imaginary, we may still take the integral in the same form, and replace the constants by their imaginary values. Putting $\alpha' = \sqrt{\frac{1}{CL} - \frac{R^2}{4L^2}}$, we get then

$$Q = Q_0 e^{-\frac{Rt}{2L}} \left[\cos \alpha' t + \frac{R}{2\alpha' L} \sin \alpha' t \right]$$

$$I = \frac{Q_0}{\alpha' LC} e^{-\frac{Rt}{2L}} \sin \alpha' t. \tag{23}$$

We get from these equations, in the two cases,

$$(24)\qquad \int_0^\infty I\,dt = Q_0, \qquad \int_0^\infty I^2\,dt = \frac{1}{2}\frac{Q_0^2}{RC}.$$

These results were evident *a priori*, for the discharge is complete, and the work it produces is reduced to a disengagement of heat; the calorific work $\int_0^\infty RI^2\,dt$ should be equal to the electrical energy (89) which the conductor had before the discharge, that is to say, $\frac{1}{2}\frac{Q^0}{C}$.

537. The nature of the discharge is very different according as the solution is given by equations (22) or equations (23).

In the first case the discharge is continuous. The strength of the current begins by being zero, passes through a maximum, and then decreases to zero. The maximum is at the time determined by the condition $\frac{dI}{dt}=0$, or

$$\left(\frac{R}{2L}-\alpha\right)e^{2\alpha t}=\frac{R}{2L}+\alpha,$$

which gives

$$t=\frac{1}{2\left(\frac{R^2}{4L^2}-\frac{1}{CL}\right)^{\frac{1}{2}}}\,l.\frac{\frac{R}{2L}+\sqrt{\frac{R^2}{4L^2}-\frac{1}{CL}}}{\frac{R}{2L}-\sqrt{\frac{R^2}{4L^2}-\frac{1}{CL}}}.$$

538. In the second case, the values of Q and of I are given by the periodic functions; the conductor takes alternately charges in contrary directions, and the wire is the seat of *alternating currents*.

The times of maxima and minima of charge correspond to $I=0$, that is to say, to

$$\sin \alpha' t = 0, \qquad \text{or} \qquad \alpha' t = n\pi.$$

The oscillations of the discharge are regular, therefore, and the value of the complete period T is

$$T=\frac{2\pi}{\alpha'}=\frac{2\pi}{\sqrt{\frac{1}{CL}-\frac{R^2}{4L^2}}}.$$

The values of the alternate maxima are

$$+Q_0,$$
$$-Q_0 e^{-\frac{R\pi}{2L\alpha'}},$$
$$+Q_0 e^{-\frac{2R\pi}{2L\alpha'}},$$
$$-Q_0 e^{-\frac{3R\pi}{2L\alpha'}}, \text{ etc.};$$

they decrease, therefore, like terms of a geometrical progression, the ratio of which is $e^{-\frac{R\pi}{2L\alpha'}}$.

The maxima of the intensity of the current in the two directions correspond to $\frac{dI}{dt}=0$, which gives

$$\tan \alpha' t=\frac{2L\alpha'}{R},$$

or

$$\sin \alpha' t=\pm \alpha'\sqrt{CL};$$

they are still equidistant, separated by a semi-period $\frac{\pi}{\alpha'}=\frac{T}{2}$, and succeed the periods at which the current is zero, by the time θ, defined by the smallest angle which satisfies the condition

$$\sin \alpha'\theta=\alpha'\sqrt{CL}.$$

The values of the maxima of intensity are successively

$$I_1=+\frac{Q_0}{\sqrt{CL}}e^{-\frac{R\theta}{2L}},$$
$$I_2=-\frac{Q}{\sqrt{CL}}e^{-\frac{R}{2L}\left(\theta+\frac{\pi}{\alpha'}\right)}=-I_1 e^{-\frac{R\pi}{2\alpha' L}},$$
$$I_3=+\frac{Q}{\sqrt{CL}}e^{-\frac{R}{2L}\left(\theta+\frac{2\pi}{\alpha'}\right)}=+I_1 e^{-\frac{2R\pi}{2\alpha' L}}, \text{ etc.};$$

they also decrease in geometrical progression.

Disregarding the sign, the total quantity of electricity put in motion is

$$(25) \qquad Q_0\left(1+2e^{-\frac{R\pi}{2L\alpha'}}+2e^{-\frac{2R\pi}{2L\alpha'}}+\ldots\ldots\right)=Q_0\frac{1+e^{-\frac{R\pi}{2L\alpha'}}}{1-e^{-\frac{R\pi}{2L\alpha'}}}.$$

This total mass is the greater the nearer the quantity $e^{-\frac{R\pi}{2L\alpha'}}$ is to unity, that is, the greater the factor $\frac{2L\alpha'}{R}$, or the greater is the ratio $\frac{L}{CR^2}$.

539. We pass to the case of a continuous current by making $Q_0=\infty$, $C=\infty$, and $\frac{Q}{C}=E$. The roots of equation (21) are then real, and the discharge is continuous; we arrive thus at the formula already found (532)

$$I=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right).$$

If we suppose the coefficient L of self-induction to be very small, we always fall into the case of real roots; the coefficient α then becomes equal to $\frac{R}{2L}\left(1-\frac{2L}{CR^2}\right)$ or $\frac{R}{2L}-\frac{1}{CR}$, and the equations (22) become

$$(26) \qquad \begin{aligned} Q&=Q_0\,e^{-\frac{t}{RC}},\\ I&=\frac{Q_0}{CR}e^{-\frac{t}{RC}}. \end{aligned}$$

The current accordingly starts at once with its maximum value, and then diminishes indefinitely.

It must, however, be remarked that all the preceding discussion rests on the implied hypothesis that electricity does not possess *inertia*, and that the intensity is the same throughout the whole extent of the wire. These hypotheses appear justified in all cases of permanent currents, or of slow variations, but we can no longer assume them in the case of sudden discharges; the results at which we have arrived can only then be considered as a first approximation.

540. Case of Two Circuits.—Let us consider two adjacent circuits C and C', whose relative position is fixed, and which contain electromotive forces; we shall represent by E, R, and L the electromotive force, the resistance, and the coefficient of self-induction of the first circuit; by the same quantities accentuated, the analogous quantities of the second; and by M the coefficient of mutual induction. All the coefficients being supposed constant, we shall have the two simultaneous equations

$$\begin{aligned} M\frac{dI'}{dt}+L\frac{dI}{dt}+RI-E&=0,\\ M\frac{dI}{dt}+L'\frac{dI'}{dt}+R'I'-E'&=0, \end{aligned} \tag{27}$$

the solution of which is of the form

$$\begin{aligned} RI-E&=Ae^{\rho t}+Be^{\rho' t},\\ R'I'-E'&=A'e^{\rho t}+B'e^{\rho' t}. \end{aligned} \tag{28}$$

The coefficients A, B, A', B' are constants to be determined from the conditions in respect to the limits.

Expressing the condition that these values of the strength satisfy the differential equations (27), whatever be the time, we find that the exponents ρ and ρ' are roots of the equation of the second degree.

$$\left(1-\frac{M^2}{LL'}\right)\rho^2+\left(\frac{R}{L}+\frac{R'}{L'}\right)\rho+\frac{RR'}{LL'}=0. \tag{29}$$

These roots are always real; again they must be negative, seeing that, under no circumstances, can the strength increase indefinitely with the time. Hence we must have $LL'>M^2$, which moreover is evident (519); we could only have $LL'=M^2$ if the two circuits coincided.

It follows also from this condition that if the coefficient of self-induction L of a wire is very small, the coefficient of mutual induction M of this wire with any other wire is also very small.

541. Consider the case in which $E'=0$, that is to say in which the second circuit contains no electromotive force.

If we only wish to determine the quantities of electricity m and m' which traverse the two circuits in time t, we may calculate the

integrals $\int I dt$ and $\int I' dt$ by equations (27). If I_0 and I'_0 are the intensities of the two currents at the beginning of the time in question, we shall have

$$(30) \qquad \begin{aligned} R\,m &= Et + L(I_0 - I) + M(I'_0 - I'), \\ R'm' &= M(I_0 - I) + L'(I'_0 - I'). \end{aligned}$$

Suppose that we close the circuit, and that we consider the phenomenon after a sufficiently long time, we shall have

$$\begin{aligned} I_0 &= 0, & I'_0 &= 0, \\ I &= \frac{E}{R}, & I' &= 0; \end{aligned}$$

and therefore,

$$(31) \qquad \begin{aligned} m &= I\left(t - \frac{L}{R}\right) = \frac{E}{R}\left(t - \frac{L}{R}\right), \\ m' &= -\frac{M}{R'}I = -\frac{E}{R}\frac{M}{R'}. \end{aligned}$$

If, on the contrary, we open the circuit C, after the steady condition has been attained, we shall have at the two limits,

$$\begin{aligned} I_0 &= \frac{E}{R}, & I'_0 &= 0, \\ I &= 0, & I' &= 0. \end{aligned}$$

The quantities of electricity induced in the two circuits are then equal, and of opposite signs to those of the preceding case—which was evident, for the variation of the flow of force was the same in both cases.

It is to be observed that the extra current of C is independent of C', and, on the other hand, that the induction on the circuit C' only depends on its resistance R', on the coefficient of mutual induction of the two currents, and on the strength I of the permanent current in the circuit C. The direct consideration of the flow of force still enables us to foresee these results.

If we wish to know the strength of the currents at each instant, the solution of the problem must be completed by determining the constants.

542. CURRENT ON OPENING.—Let us first consider the case in which, having closed the current in the principal circuit C, connection with the battery is broken by leaving the circuit open; an induced current is formed in the secondary circuit C', but the principal current is *entirely* broken. The first of the equations (27) no longer holds, and the second reduces to

$$L'\frac{dI'}{dt}+R'I'=0;$$

it is identical with equation (9), and therefore the law of the extra current will be the same in both cases.

In like manner, on the hypothesis that the opening of the circuit C was instantaneous, and that the current is not prolonged with a variable resistance, as in the case in which a spark is produced, we may determine the initial value of the current produced in C'.

Let us integrate the second of these equations (27), taking $E'=0$, from $t=0$, to a time τ which is infinitely small in comparison with the duration of the induction current in C'; denoting by I'_1 the intensity of the current induced at the time τ, we shall have the equation

$$-M\frac{E}{R}+L'I'_1+R'\int_0^{\tau}I'dt=0. \tag{32}$$

If we make τ diminish towards zero, the last term tends itself to zero, seeing that τ has an infinitely small value, and that the intensity I' of the current retains a finate value; we have then, in the limit,

$$I'_1=\frac{ME}{L'R}=\frac{M}{L'}I_0. \tag{33}$$

Thus the initial intensity of the induced current is to the intensity of the inducing current I_0 in the ratio of the coefficients M and L'.

Hence, at any given instant, we shall have, by equation (10),

$$I'=\frac{ME}{L'R}e^{-\frac{R't}{L'}}. \tag{34}$$

From this is deduced, for the quantity of electricity put in motion,

$$\int_0^\infty I'dt = \frac{ME}{L'R}\frac{L'}{R'} = \frac{E}{R}\frac{M}{R'},$$

and, for the calorific work expended in the circuit,

$$R\int_0^\infty I'^2 dt = \frac{1}{2}\frac{E^2}{R^2}\frac{M^2}{L'} = \frac{1}{2}L'I_1'^2.$$

543. CURRENT ON CLOSING.—The moment the inducing current is closed, the two circuits react on each other, and we must take into account the two simultaneous equations (27).

If we put

$$\alpha = \sqrt{\left(\frac{RL' - R'L}{2RM}\right)^2 + \frac{R'}{R}}, \tag{35}$$

the roots of the equation are

$$\rho' = \frac{-(RL' + R'L) - 2RM\alpha}{2(LL' - M^2)},$$

$$\rho = \frac{-(RL' + R'L) + 2RM\alpha}{2(LL' - M^2)}.$$

The coefficients will be determined by the aid of equations (27) and (28) by the condition that, for $t = 0$, we have $I = 0$ and $I' = 0$, which gives

$$\frac{M}{R'}(A'\rho + B'\rho') + \frac{L}{R}(A\rho + B\rho') = E,$$

$$\frac{M}{R}(A\rho + B\rho') + \frac{L'}{R'}(A'\rho + B'\rho') = 0,$$

$$A + B + E = 0,$$

$$A' + B' = 0.$$

We get then

$$I = \frac{E}{R}\left\{1 - \frac{1}{2}\left[\left(1 + \frac{R'L - RL'}{2RM\alpha}\right)e^{\rho t} - \left(1 - \frac{R'L - RL'}{2MR\alpha}\right)e^{\rho' t}\right]\right\},$$

$$I' = -\frac{E}{2R\alpha}\left(e^{\rho t} - e^{\rho' t}\right). \tag{36}$$

The differential of the latter equation

$$\frac{dI'}{dt} = -\frac{E}{2Ra}\left(\rho e^{\rho t} - \rho' e^{\rho' t}\right)$$

shows that, in the secondary circuit C', the induced current, which is always negative, since we have $\rho' > \rho$ in absolute value, starts from zero, without its initial differential being zero, passes through a maximum, and then decreases to zero.

It will be seen that the inducing current commencing at zero, increases progressively to its maximum value $\frac{E}{R}$ relative to the permanent state.

If we suppose the two circuits C and C' identical, the formulæ reduce to

$$(37)\qquad \begin{aligned} I &= \frac{E}{R}\left[1 - \frac{1}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right)\right], \\ I' &= -\frac{E}{2R}\left[e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right]. \end{aligned}$$

If, further, the two circuits are in contact, the coefficients L and M are very slightly different, and we have sensibly

$$(38)\qquad \begin{aligned} I &= \frac{E}{R}\left[1 - \frac{1}{2}e^{-\frac{Rt}{2L}}\right], \\ I' &= -\frac{E}{2R}e^{-\frac{Rt}{2L}}. \end{aligned}$$

In this latter case the strength of the direct current produced on opening the circuit (533), is

$$(39)\qquad I_1' = \frac{E}{R}e^{-\frac{Rt}{2L}}.$$

The two currents attain their maximum, for $t = 0$, and this maximum is twice as great for the direct as for the inverse current.

544. Two Circuits with Variable Electromotive Force.— Suppose we have

$$E = E_0 \sin\frac{2\pi}{T}t,$$

and that the second circuit is closed without containing any electromotive force.

When the permanent state is attained, the two currents are still periodic, and as in (355), we may write

$$\text{(40)} \qquad \begin{aligned} I &= A \sin 2\pi \left(\frac{t}{T} - \phi\right), \\ I' &= A' \sin 2\pi \left(\frac{t}{T} - \phi'\right). \end{aligned}$$

Putting

$$r = R + \frac{4\pi^2}{T^2} \frac{M^2 R'}{R'^2 + \frac{4\pi^2}{T^2} L'^2},$$

$$l = L - \frac{4\pi^2}{T^2} \frac{M^2 L'}{R'^2 + \frac{4\pi^2}{T^2} L'^2},$$

the condition that the differential equations are satisfied for any given time by their values of E, I, and I', gives

$$\text{(41)} \qquad \begin{aligned} A^2 &= \frac{E_0^2}{r^2 + \frac{4\pi^2 l^2}{T^2}}, \\ \tan 2\pi\phi &= \frac{2\pi l}{T} \frac{}{r}. \end{aligned}$$

These expressions have the same form as above (535). It will be seen that the effect of the presence of the second circuit is to increase the apparent resistance of the first, and to diminish the apparent coefficient of self-induction.

We shall find, in like manner, for the second circuit,

$$\text{(42)} \qquad \begin{aligned} A'^2 &= A^2 \frac{4\pi^2}{T^2} \frac{M^2}{R'^2 + \frac{4\pi^2}{T^2} L'^2}, \\ \tan 2\pi\phi' &= \frac{T}{2\pi} \frac{\frac{4\pi^2}{T^2} L'l - R'r}{L'r + R'l}. \end{aligned}$$

The amplitude A′ is at first inversely as the period, so long as the oscillations are not rapid; for a very short period we shall have

$$\frac{A'}{A}=\frac{M}{L'}.$$

This problem corresponds to the case of an induction coil, the inducing current of which was sinusoidal.

545. TELEPHONE AND MICROPHONE.—For a circuit placed in a variable magnetic field, equation (10) of (518) becomes

$$\frac{dQ}{dt}+L\frac{dI}{dt}+I\frac{dL}{dt}+RI-E=0. \tag{43}$$

Let us suppose that the circuit contains a constant electromotive force, that the factor L is constant, and that the coefficient Q varies periodically. This is the case, for instance, of an electromagnet, in front of which a magnet is made to oscillate. We may write, in that case,

$$Q=Q_0+Q_1\sin\frac{2\pi t}{T}.$$

The result is the same as if the electromotive force of the battery experienced a periodical variation equal to $Q_1\frac{2\pi}{T}\cos 2\pi\frac{t}{T}$.

When the permanent state is attained, the induced current oscillates according to the same period, and may still be represented by the expression

$$I=I_0+A\sin 2\pi\left(\frac{t}{T}-\phi\right).$$

Substituting this value in equation (43), we deduce from it

$$\tan 2\pi\phi=-\frac{2\pi L}{TR},$$

$$A^2=\frac{4\pi^2}{T^2}\frac{Q_1^2}{R^2+\frac{4\pi^2L^2}{T^2}}.$$

If the electromotive force E is zero, the current transmitted is periodical; this is the case of the telephone.

All the other quantities being constant, it may happen that the resistance R varies periodically; the current will still be periodical, and the amplitude of the variations will be proportional to the strength of the current which is produced with a constant resistance—that is to say proportional to the electromotive force.

Such a current will produce periodical induced currents in an adjacent circuit. This is what takes place in the *microphone* when the variations in resistance are produced by the vibrations of two bodies in contact, such as pieces of carbon; it is also the case with the selenium *photophone*, in which the variations in resistance produced by the intermittent action of light are utilised.

The same result would be attained if the coefficient of self-induction L was variable, which might be attained either by periodically altering the configuration of the circuit, or by oscillating the iron armature of an electromagnet.

546. Induction in an Open Circuit.—We have hitherto only considered closed circuits. If the wire submitted to induction is not closed, we may assume that the effect of the electromotive forces of induction is to tend to move the electrical masses towards the end of the wire, and to set up for a moment a definite difference of potential between these ends. On Maxwell's theory of displacement this problem does not present any fresh difficulties, for the circuits are always closed by the dielectric. It seems natural, moreover, to extend the theory to the case of open circuits, especially if the circuit contains a great number of turns (as in the case of coils), and if the conditions are such that the current could at each instant be regarded as identical throughout the whole length of the wire.

Let us consider, for instance, the induction of two adjacent circuits (540), and suppose that the ends of the induced wire communicate separately with the armatures of a condenser of capacity C. If we open the inducing circuit, and call V the difference of potential of the armatures of the condenser at the period t after this break, the intensity I′ of the induced current will be defined by the equations

$$\text{(44)} \qquad \begin{aligned} L'\frac{dI'}{dt} + R'I' + V &= 0, \\ I' &= C\frac{dV}{dt}. \end{aligned}$$

We may further suppose that the insulation of the condenser

is not absolute, and that the resistance of the dielectric, instead of being infinite, is represented by R_1. The latter equation should then be replaced by

$$I = \frac{V}{R_1} + C\frac{dV}{dt},$$

from which follows

$$\frac{d^2V}{dt^2} + \left(\frac{1}{CR_1} + \frac{R'}{L'}\right)\frac{dV}{dt} + \frac{1}{L'C}\left(\frac{R'}{R_1} + 1\right)V = 0.$$

This equation is the same as equation (19), and has an integral of the form $V = Ae^{\rho t} + A'e^{\rho' t}$. If the roots of the quadratic equation analogous to equation (21), which determines the coefficients ρ and ρ', are imaginary, as, for the most part, will be the case unless the resistance R_1 of the dielectric is very small, the potential V is represented by a periodical function of decreasing amplitude. The value of the period is (538)

$$T = \frac{2\pi}{a'} = \frac{2\pi}{\sqrt{\frac{1}{CL'}\left(\frac{R'}{R_1} + 1\right) - \frac{1}{4}\left(\frac{1}{CR_1} + \frac{R'}{L'}\right)^2}}.$$

The constants of the general integral will be defined by the condition that for $t = 0$ we have $V = 0$ and $I' = I'_1$. Putting $a = \frac{1}{CR_1} + \frac{R'}{L'}$, we thus get

$$V = \frac{I'_1}{Ca'} e^{-\frac{at}{2}} \sin a't.$$

On the other hand, the initial intensity I'_1 of the induced current is given as a function of the inducing current I_0 (542) by an equation (33) in which the resistance of the circuit does not enter. We get then, by substitution,

(45)
$$V = I_0 \frac{M}{CL'a'} e^{-\frac{at}{2}} \sin a't,$$
$$V = I_0 \frac{M}{CL'} \frac{T}{2\pi} e^{-\frac{at}{2}} \sin \frac{2\pi t}{T}.$$

We may control this result by connecting the ends of the wire at a given period, and for a very short time, with the binding screws

of an electrometer, which determines the difference of potential. Experiment has shown that the phenomena are in complete agreement with this theory.

If we detach the condenser at the ends of the wire, the electricity set in motion by the electromotive force of induction charges the wire throughout its entire extent, especially near the ends, and the intensity of the current is not at each moment the same throughout the entire length of the wire. It may also happen that the electricity thus accumulated on the surface of the wire escapes across the surrounding medium by a series of branch circuits. These two circumstances complicate the problem, and the preceding equations no longer present an exact solution, but the difference of potentials at the end of the wire is in most cases still represented by a periodical oscillation of decreasing amplitude.

547. Laws of Branch Currents in the Variable State.— The law which in a closed polygon connects the strengths of the currents with the electromotive forces (211), also applies in the variable state, provided that to the ordinary electromotive forces we add those arising from the effects of induction.

We shall first examine the case in which the current bifurcates between two points A and B, along conductors r and r', which contain no electromotive forces. To define our ideas, we shall assume that these conductors are wound as coils, and we shall call L, L', and M their coefficients of induction. If, at a given time, we denote by i and i' the strengths of the currents in the two branches, we have

$$ri+\frac{d}{dt}(\mathrm{L}-\mathrm{M})i=r'i'+\frac{d}{dt}(\mathrm{L}'-\mathrm{M})i'. \tag{46}$$

Let us first consider the case in which the second wire is uncoiled; the coefficient L' is very small, as is also the coefficient of mutual induction M (540). Equation (44) reduces to

$$ri-r'i''+\mathrm{L}\frac{di}{dt}=0. \tag{47}$$

As long as the general current is increasing, we see that the strength i' in the rectilinear branch is greater than it would be in the permanent state. The branch which contains a coil has therefore an *apparent* resistance which is greater than its *real* resistance. The difference is greater the greater is the coefficient

L, and the more rapid is the variation in strength; when the strength is decreasing the reverse is the case. The effect is the same in the general case, in which none of the coefficients is zero, and equation (46) shows that each conductor behaves for an increasing current, as if it had a greater resistance than for the permanent state.

Let m and m' be the quantities of electricity which traverse the two conductors in the time t, we have

$$rm - r'm' + \left[Li - L'i' + M(i' - i)\right]_0^t = 0.$$

In the case of a discharge, the initial and the final strengths are zero; if the coefficients L, L', and M were constant, the term within the parenthesis is null, and we have

$$rm - r'm' = 0.$$

It follows that the total quantities of electricity were divided between the two branches according to the ordinary law, although at each instant the division was different; there is therefore ultimately compensation.

This conclusion is only valid when the two circuits have produced no external work; and particularly that during the discharge no magnet shall have been moved near one of them. Hence, in measuring discharges, it is difficult to use a galvanometer with a shunt.

548. If the principal current is sinusoidal with the amplitude I_0, the two branch currents will ultimately have the same periodic character with an unequal difference of phase. By calculations which are analogous to the foregoing, we shall find the following relations between the amplitudes A and A', and the difference of phases ϕ and ϕ',

$$\frac{A^2}{r'^2 + \frac{4\pi^2}{T^2}(L' - M)^2} = \frac{A'^2}{r^2 + \frac{4\pi^2}{T^2}(L - M)^2} = \frac{I_0^2}{(r + r')^2 + \frac{4\pi^2}{T^2}(L + L' - 2M)^2},$$

$$\tan 2\pi\phi = \frac{2\pi}{T} \frac{Lr' - L'r + M(r - r')}{(r + r')r' + \frac{4\pi^2}{T^2}(L + L' - 2M)(L' - M)},$$

$$\tan 2\pi\phi' = \frac{2\pi}{T} \frac{(L'r - Lr' + M(r' - r)}{(r + r')r + \frac{4\pi^2}{T^2}(L + L' - 2M)(L - M)}.$$

There is evidently a difference of phase between the principal current and the sinusoidal electromotive force which produces it.

549. Let us further consider as an application the experiment by which Faraday proved the existence of the direct extra current which is produced when the circuit is broken. A battery is closed by a circuit with two branches, like that which we have been considering; one of the wires, of the resistance r', is rectilinear, the other, of the resistance r, consists of a coil formed of a great number of turns.

If E is the electromotive force of the battery, R the resistance, and I the principal current, after closing the circuit we shall have the equations

$$\begin{aligned} &I = i + i', \\ &E = RI + r'i', \\ &L\frac{di}{dt} + ri = r'i'. \end{aligned} \tag{48}$$

Putting

$$\rho^2 = Rr + Rr' + rr',$$

and observing that the currents are zero for $t = 0$, we deduce for the current which traverses the coil at the period t,

$$i = \frac{r'E}{\rho^2}\left(1 - e^{-\frac{\rho^2 t}{L(R+r')}}\right),$$

and for that which traverses the rectilinear wire,

$$i' = \frac{rE}{\rho^2}\left(1 + \frac{Rr'}{(R+r')r}e^{-\frac{\rho^2 t}{L(R+r')}}\right).$$

If, when once the steady condition is established, we break the principal circuit, so that the spark on breaking has no appreciable duration, making $R = \infty$, equations (47) give

$$i + i' = 0,$$

and therefore

$$L\frac{di}{dt} + (r + r')i = 0.$$

The integral of this equation determines the induced current which traverses the two branches. Observing that the current i is defined by the permanent state, we shall have for the time t' after opening the circuit,

$$i = \frac{Er'}{\rho^2} e^{-\frac{(r+r')t'}{L}}.$$

This current in the rectilinear branch is in the opposite direction to the primitive current.

The total quantity of induced electricity is

$$m = \frac{EL}{\rho^2} \frac{r'}{r+r'}.$$

If the needle of a galvanometer placed on a rectilinear wire abuts against an obstacle which prevents it from obeying the permanent current, this needle will be driven in the opposite direction by the induction current, when the principal circuit is opened. This method was used by Faraday.

550. If we regard the problem in all its most general forms, we may propose to determine the division of the currents in the cases in which the conductor is made up of a network of wires forming polygons of any given form.

Let us suppose that a portion of the network is made up of a polygon whose sides are $A_1, A_2, A_3 \ldots$. Let $i_1, i_2, i_3 \ldots$ be the intensities at a given moment of the currents which traverse the various branches; $r_1, r_2, r_3 \ldots$ their resistances; E_1, E_2, E_3 the electromotive forces contained in each of them; $L_1, L_2, L_3 \ldots$ their coefficients of self-induction; $M_{1.2}, M_{1.3}, M_{2.3} \ldots$ the coefficients of mutual induction of each branch on another part of the circuit, in which the intensity of the current is i'. For each side of the polygon we have an equation, such as the following, relative to the side A_1:

$$E = i_1 r_1 + \frac{d}{dt}\left[i i_1 + M_{1.2} i_2 + M_{1.3} i_3 + \ldots \Sigma M'_1 i' \right],$$

from which is deduced for the contour of this polygon,

$$\Sigma E = \Sigma ir + \Sigma \frac{d}{dt}\left[Li + \Sigma Mi + \Sigma M' i' \right].$$

The expression $\Sigma \mathrm{M}i$ represents the sum of the products of each coefficient of mutual induction by the sum of the intensities of the current in the two corresponding wires. If there is a magnetic or electrical system external to the first, the corresponding variation of the flow of magnetic force Q in the polygon will also produce an electromotive force. We shall thus have for the contour of the triangle,

$$(48) \qquad \Sigma \mathrm{E} = \Sigma ir + \Sigma \frac{d}{dt}\left[\mathrm{L}i + \Sigma mi + \Sigma m'i'' \right] + \frac{d\mathrm{Q}}{dt}.$$

Integrating this expression between 0 and t, we get

$$\Sigma \int \mathrm{E}dt = \Sigma \int irdt + \Sigma \left[\mathrm{L}i + \Sigma \mathrm{M}i + \Sigma \mathrm{M}'i'' \right]_0^t + \left(\mathrm{Q}_t - \mathrm{Q}_0 \right).$$

If all the coefficients of induction are constant, and the initial and final intensities are the same in each of the wires, the second term of the second member is null. Finally, if the external work is also null, we get simply

$$\Sigma \int \mathrm{E}dt = \Sigma \int irdt,$$

or if m is the quantity of electricity which passes in a wire of resistance r, and supposing the resistances constant,

$$\Sigma \int \mathrm{E}dt = \Sigma mr.$$

Comparing this equation with that given by Kirchhoff's law (211), we see that they are of the same form, and we may deduce from it the following theorems:

1st. The distribution of the discharges in any given circuit is inversely as the resistances, when the form of the circuit does not change, and the initial and final intensities are the same in each wire, and there is no external work.

2nd. Even when the former conditions are not fulfilled for all the wires, the equations of distribution do not contain the coefficients of self-induction of the branches in which the final and initial intensities are the same.

551. Phenomena of Induction in Telegraph Cables.—In studying the propagation of electricity in cylindrical conductors

during the variable state, we have disregarded the effects of induction, which are due to changes in the strength of the current. From this there arises a fresh cause of delay in establishing and in suppressing the principal current, and this retardation can no longer be calculated as we have done before (535), since the duration of the propagation is very considerable as compared with the duration of the phenomena of induction; it is not possible to assume then that the strength of the current has the same value at every instant in the whole extent of the circuit.

If there are a series of alternate charges and discharges in the wire, as is the case with telegraphic signals, we may consider the phenomenon as being due to a variable electromotive force, and effects will be produced analogous to those which have been mentioned above.

The cable may, moreover, be considered as near other ones, which react on the first either by their mere presence, or by the variations in their own currents which traverse them. The resultant effects are very manifest in air wires resting on insulators.

Finally, when several conductors are enclosed in the same dielectric sheath, as is the case with subterranean, or submarine cables, the potential at each point of one of the wires depends on the charge of the adjacent wires. From this follows a new kind of influence or induction, purely electrostatic in character, and which Sir W. Thomson called *peristaltic*, to distinguish it from that of Faraday. This phenomenon is completely analogous with the reciprocal influence of elastic tubes connected lengthwise, which are filled and surrounded by the same liquid, when the liquid is made to circulate in one or more of the tubes, while the ends of the others are opened or closed, or in any other special condition; a closed tube would correspond to an insulated conducting wire, and an open tube to an uninsulated wire.

From these suggestions it will be seen how complex is the problem of the propagation of electricity, if we take into account all the circumstances which may have an influence on the phenomenon.

552. CALCULATION OF THE COEFFICIENTS OF INDUCTION.—SOLENOIDS.—The preceding examples show the importance of the part which the coefficients M and L play in the calculation of the phenomena of induction. We shall examine a few simple cases, in which these coefficients may be easily estimated.

Let us, in the first case, consider a cylindrical solenoid so long that for a considerable portion of its length we may disregard the action of the ends.

If S is the section of the cylinder, which we will assume to be circular, I the strength of the current, and n_1 the number of turns for unit length, the flow of force or of magnetic induction is equal to $4\pi n_1 IS$ (495); and, when the current is equal to unity, the value of this flow is

$$g = 4\pi n_1 S.$$

Suppose that n' additional turns of any given diameter are wound on the cylinder, the flow of force of the first circuit traverses n' times the surface of the second; the coefficient of mutual induction is then

$$M = n'g = 4\pi n_1 n' S. \tag{49}$$

The same flow traverses the surface of the first circuit n_1 times for unit length, so that the coefficient of self-induction of the solenoid for unit length, is

$$L_1 = n_1 g = 4\pi n_1^2 S. \tag{50}$$

The values found for the coefficients M and L_1 are a maximum, for the true flow of force is less than $4\pi n_1 IS$, and this flow diminishes as we come near the ends of the solenoid, where it becomes even less than $2\pi n_1 IS$. For the magnetic induction in the uniformly magnetised cylinder, which is the equivalent of a solenoid (373), is equal to the resultant of the force $4\pi n_1 I$ parallel to the axis, and of the action of the two terminal layers of uniform densities $\pm\sigma = n_1 I$, which is in the contrary direction. But in a section near the positive surface, this surface sends towards the interior a flow of force equal to $2\pi n_1 IS$, to which must be added, in order to obtain the true flow, the portion of the flow from the negative surface which traverses the same edge.

Suppose that the n' turns in question form a solenoid, concentric with the first and of the same length l. Let us denote by r_1 and r_2 the two radii, and by n_1 and n_2 the number of turns in unit length of each of them. Disregarding the action of the ends, we may write

$$M = n_2 lg = Ng = 4\pi n_1 n_2 lS,$$

putting

$$N = n_2 l,$$
$$g = 4\pi n_1 S = 4\pi^2 n_1 r_1^2.$$

The factor N represents the total number of turns of the outer solenoid, and g the flow of force produced in the inner solenoid by a current equal to unity.

553. CONCENTRIC COILS.—Let us in like manner consider two concentric coils composed of several layers of wire closely pressed together, so that in unit surface of a meridian plane, the number of wires is respectively n_1^2 and n_2. For layers of the thicknesses dr_1 and dr_2, and of lengths equal to unity, the number of wires will in like manner be $n_1^2 dr_1$, and $n_2^2 dr_2$. We shall have then

$$N = l n_2^2 \int dr_2,$$

$$g = 4\pi^2 n_1^2 \int r_1^2 dr_1,$$

for each coil, the integral being extended to the entire thickness. Denoting the extreme radii by y_1 and x_1 for the first coil, and by y_2 and x_2 for the second, we get

$$N = n_2^2 l (y_2 - x_2),$$

$$g = \frac{4\pi^2 n_1^2}{3} (y_1^3 - x_1^3).$$

If we disregard the effect of the ends, the value of the coefficient of mutual induction will be

$$(51) \qquad M = Ng = \frac{4}{3} \pi^2 n_1^2 n_2^2 l (y_2 - x_2)(y_1^3 - x_1^3).$$

When the layers of the two coils are in contact, we may put $x_2 = y_1 = y$. If we wish to arrange the intermediate radius so that the coefficient M is a maximum, we must satisfy the condition

$$\frac{y_2 - y}{y - x_1} = \frac{y^3 - x_1^3}{3y^2(y - x_1)} = \frac{1}{3}\left(1 + \frac{x_1}{y} + \frac{x_1^2}{y^2}\right).$$

Since the latter member of this equation is less than unity, the outer coil must not be so thick as the inner one.

In order to obtain the coefficient of self-induction of a coil, we may assume, for instance, that two identical coils are superposed, and we may determine what is then their coefficient of mutual induction (512). If in this way we make in the preceding equations

$$n_1 = n_2,$$
$$y_1 = y_2 = y,$$
$$x_1 = x_2 = x,$$

we get

$$(52) \qquad L = \frac{4}{3}\pi^2 n_1^4 l (y-x)(y^3 - x^3).$$

554. Coils with a Soft Iron Core.—Let us assume that the inner coil contains a cylindrical core of soft iron of radius a, the coefficient of magnetisation of which is k. The value of N does not change; but so long at least as we remain within the limits within which magnetisation is proportional to the magnetising force, the magnetic induction in the space occupied by the soft iron is equal (379) to the original value of the force multiplied by $1 + 4\pi k$. In the case of a solenoid the value of g then becomes

$$g = 4\pi n_1 (S + 4\pi k \pi a^2) = 4\pi^2 n_1 (r_1^2 + 4\pi k a^2).$$

If the coil consists of several layers, we have

$$g = 4\pi^2 n_1^2 \int (r_1^2 + 4\pi k a^2)\, dr_1 = 4\pi^2 n_1^2 \left[\frac{y_1^3 - x_1^3}{3} + 4\pi k a^2 (y_1 - x_1) \right];$$

it follows that

$$(53) \qquad M = \frac{4}{3}\pi^2 n_1^2 n_2^2 l (y_2 - x_2) \left[y_1^3 - x_1^3 + 12\pi k a^2 (y_1 - x_1) \right].$$

When the iron core entirely fills the inner cavity, and the two coils are in contact, we may put

$$a = x_1 = x,$$
$$x_2 = y_1 = y,$$
$$y_2 = z;$$

we get then

$$(54) \qquad M = \frac{4}{3}\pi^2 n_1^2 n_2^2 l (z - y)(y - x) \left[y^2 + xy + x^2 + 12\pi k x^2 \right].$$

If the radius of the core is the only variable, and we wish to arrange it so that the coefficient of mutual induction is a maximum, the partial differential of M in respect of x should be null; it follows that

$$x = y \frac{8\pi k}{1 + 12\pi k},$$

or sensibly $3x = 2y$, since the coefficient of magnetisation k is very great.

If the external radius z is given, as well as the way in which the wires are coiled, and we wish to arrange x and y so that M is a maximum, making the partial differentials $\frac{\partial M}{\partial x}$ and $\frac{\partial M}{\partial y}$ equal to zero, we shall find in like manner

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{4}.$$

Lastly, the coefficient of self-induction of a coil which has the external radius z, and the internal radius y, and which includes a soft iron core of radius x, will be expressed by

$$(55) \qquad L = \frac{4}{3}\pi^2 n_1^4 l (z-y)^2 \left[z^2 + zy + y^2 + 12\pi k x^2 \right].$$

The problem we have here discussed is that which occurs in the construction of induction coils.

555. ANNULAR COILS.—In a coil formed by regularly winding a wire on a circular ring, the coefficient g (496) becomes

$$g = 4\pi n_1 \int \frac{dS}{x}.$$

Suppose that a second wire is coiled n' times round the first in any way whatever; the coefficient of mutual induction will be

$$(56) \qquad M = 4\pi n_1 n' \int \frac{dS}{x}.$$

If the first coil contains a soft iron core of revolution about the same axis, and of section S′, the value of g is then

$$g = 4\pi n_1 \left[\int \frac{dS}{x} + 4\pi k \int \frac{dS'}{x} \right].$$

When the core fills the whole cavity of the solenoid this value reduces to

$$g = 4\pi n_1 (1 + 4\pi k) \int \frac{dS}{x}.$$

In this case the coefficient of mutual induction on an external wire which makes n' turns, is

$$M = 4\pi n_1 n' (1 + 4\pi k) \int \frac{dS}{x}, \tag{57}$$

and the coefficient of self-induction of the coil itself, which contains $2\pi n$ turns, is

$$L = 8\pi^2 n_1^2 (1 + 4\pi k) \int \frac{dS}{x} \tag{58}$$

We have seen (502) what are the values of the integral $\int \frac{dS}{x}$ in certain special cases.

556. ELECTRICAL MOTORS. — *Electrical motors* are machines containing conducting wires, electromagnets, or permanent magnets, and so arranged that when a current produced by an extraneous source is introduced, the electromagnetic or electrodynamic actions, exerted between the different parts, are used to produce a relative displacement of these parts. The continuity of the motion is obtained either by sliding contacts, or by commutators which alter the direction of the current in the machine.

These machines may be so made as to work uniformly when the strength of the current is constant, as with Faraday's disc (530); but more frequently the actions are periodic, and the relative motion of the parts, whether oscillating or rotating, produces an electromotive force of induction E of a periodic character; in

order that the motion may be kept up, the current in one part of the circuit must be reversed by a commutator twice during each period, so that the electromotive force of induction be at each instant in the opposite direction to E_0—that of the exciting current.

When a regular state is attained, the work of the reciprocal actions is entirely consumed in overcoming external resistances, for the velocity resumes the same value at the end of each period.

The energy expended by the source, during each period θ, is equal to the external work increased by the energy which corresponds to the heating of the conductors. If R is the resistance of the circuit, we have then,

$$(59) \qquad \int_0^\theta E_0 I\,dt = \int_0^\theta I^2 R\,dt + \int_0^\theta EI\,dt.$$

The reversals of the current by the commutator produce sparks which absorb part of the work, and which affect the equation if we take count of the corresponding variations in the resistance R of the circuit.

In the general case the values of the integrals depend on the law according to which the current varies during a period; but if the strength is sensibly uniform, equation (59) reduces to

$$(60) \qquad E_0 - E = IR.$$

The efficiency ρ of such a motor is the quotient of the external work EI in unit time, by the total energy expended $E_0 I$, or the ratio $\frac{E}{E_0}$ of the electromotive forces. If I_0 is the strength of the current which the electromotive force E_0 would produce in the circuit at rest, we have

$$(61) \qquad \rho = \frac{E}{E_0} = 1 - \frac{IR}{E_0} = 1 - \frac{I}{I_0}.$$

The expression for the external work itself is

$$(62) \qquad EI = \frac{E(E_0 - E)}{R} = RI(I_0 - I).$$

As the external electromotive force is constant, this work is a maximum when the motion reduces the strength of the current by one-half, and the efficiency is then equal to 0.50.

If the external work is very small, the velocity of the motor increases very rapidly with or without limit according to the mode of construction; the electromotive force of induction tends to become equal to the external electromotive force, and the efficiency tends towards unity.

557. ELECTROMOTORS.—When an electrical motor is set in motion by an extraneous machine, it becomes the seat of an electromotive force, opposite in sign to that which would produce the motion, and the circuit which constitutes it is, in general, traversed by an electrical current. The apparatus is then a producer of electricity, or an *electromotor*.

Suppose that owing to any temporary cause there is in the circuit a current of the strength i; if the work Ei, absorbed by the electromotive force of induction, is greater than the thermal energy disengaged in the conductors, the current will go on increasing until

$$EI = I^2R. \tag{63}$$

If the condition $E > iR$ holds for an infinitely small current, the machine once in motion will prime itself, and for a steady condition will produce a current defined by the preceding equation (61). When, on the contrary, $E < iR$ for an infinitely small current, the external work cannot create and maintain an electrical current, unless there is artificially introduced into the circuit a current of such strength that the condition $E \geqq iR$ is realised, after which the extraneous electromotive force could be suppressed without the current ceasing.

A machine used as an electromotor is therefore capable, or not, of creating an electric current, or of keeping up a current already established, according to the value of the total resistance, or, what produces an analogous effect, according to the kind of external work which the current is to produce. Since the electromotive force of induction, other things being equal, is proportional to the velocity of the machine, we see that for a given total resistance, and given external work, the electromotor will be able to produce and maintain a current the more rapidly, the greater is its velocity.

Let us consider two extreme cases:

1st. If a machine consists of permanent magnets which produce an invariable field in which the conducting wires move (take

Faraday's disc, for instance), the electromotive force is simply proportional to the number n of turns, or of oscillations of the machine in unit time, and may be represented by nE_1. When the velocity is constant, such a machine acts exactly like an ordinary battery. It can always produce a current in a metallic circuit, since the condition $E > iR$ always holds when the current is very feeble.

2nd. If the machine is composed of fixed wires and of movable wires, or of two systems of electromagnets (provided that we remain within the limits in which the magnetisation is proportional to the magnetising force), the electromotive force is proportional to the strength of the current, and may be represented by nAI. A machine of this kind can only produce and keep up a current when the velocity is so great that $nA > R$. For any value of n greater than $\frac{R}{A}$, the strength of the current increases until the resistance of the circuit, owing to the heat disengaged, has reached the value nA.

Suppose that the current has to overcome an electromotive force E′ external to itself.

The value of the efficiency of the apparatus, as in the case of the battery, is

$$\rho = \frac{E'}{E} = 1 - \frac{IR}{E},$$

and the work utilised in unit time is

$$E'I = \frac{E'(E - E')}{R}.$$

In the former case the conditions of efficiency, and of maximum work, are the same as for a battery.

In the second case, in which $E = nAI$, we have

$$\rho = 1 - \frac{R}{nA},$$

$$EI' = (nA - R)I^2.$$

The efficiency is constant, and the work utilised is proportional to the square of the current.

If the external work is nothing more than that of setting in motion a second machine identical with the first, the electromotive forces E and E′ are proportional to the number of turns n and n' of the two machines; on the other hand, these electromotive forces

are proportional to the same function $\phi(I)$ of the intensity. The efficiency

$$\rho = \frac{n'\phi(I)}{n\phi(I)} = \frac{n'}{n}$$

is about equal to the ratio of the velocities of the two machines, and the useful work in unit time is expressed by

$$E'I = \frac{n'(n-n')}{R}\left\{\phi(I)\right\}^2.$$

As the strength I is a function of $n - n'$, the maximum useful effect can only be determined if we know the form of the function ϕ.

558. The preceding considerations only apply rigorously to the case of a uniform current like that obtained with a machine such as Faraday's disc. In a periodical machine, for instance, a frame turning with a uniform motion about a vertical axis under the influence of the terrestrial field, the phenomena are more complicated.

If the machine is used as an electromotor, the ends of the wire being joined by sliding contacts with an external conductor in such a way that the coefficient of mutual induction of the two parts of the circuit may be neglected, the current should satisfy equation (518)

$$L\frac{dI}{dt} + RI + \frac{dQ}{dt} = 0,$$

in which L is the coefficient of self-induction of the whole circuit, R the resistance, and Q the flow of terrestrial force across the frame. If S is the surface of the frame, H the horizontal component of the terrestrial field, and T the period of rotation, we may write

$$Q = HS \sin 2\pi \frac{t}{p}.$$

The electromotive force is sinusoidal, like that which we investigated in (535). When the permanent régime has been established, the current is itself periodic, and the work necessary to keep up the motion corresponds altogether to the heat disengaged in the circuit, which for unit time gives

$$W_1 = \frac{R}{2} \cdot \frac{H^2S^2}{R^2 + \frac{4\pi^2L^2}{T^2}}.$$

Instead of sending the alternating current into the external circuit, a commutator may be placed on the axis of rotation, consisting of two half discs communicating separately with the ends of the frame, and of two springs connected with the external circuit. If the springs are so adjusted as to change the connections the moment the current is null—that is to say, at a period ϕT after the passage of the frame through a plane perpendicular to the magnetic meridian—the direction of the current in the external circuit is always the same, and there is no change in the law of the phenomena, as the commutator does not cause any loss of energy.

If we use the frame as a motor by connecting it with an external electromotive force E_0, it will be necessary to use a commutator which changes the direction of the current twice in each period, in order to maintain the motion. There is then a loss of energy by the sparks at the movable contacts which must be allowed for, and the calculations are far less simple.

It is impossible to treat here in greater detail the question of induction-electromotors. We shall return to the subject in the second part of this work.

559. APPLICATION TO THE STUDY OF MAGNETISM.—We are now in a position to justify the method mentioned in (417) for studying the distribution of magnetism, a method employed in the experiments of Van Rees and of Gaugain.

If a magnetised bar is surrounded at a point by a coil formed of n' turns, connected with a galvanometer, and if the coil is suddenly made to slide through a certain distance, parallel to the bar, coming to rest in its fresh position, a certain quantity of electricity m passes through the coil; if R is the total resistance of the circuit, the expression $\frac{m\mathrm{R}}{n'}$ is equal to the flow of magnetic force, which proceeds from the magnet between the two positions of the coil. Working in this way by successive displacements, we can determine the law of variation of the flow of the lateral force.

If the coil is at first in the middle of the magnet, or more exactly at the neutral point, and it is suddenly removed to a great distance, we shall get the total flow of force from the magnet, and therefore the total mass of free magnetism contained in the corresponding portion of the bar.

If the auxiliary coil thus surrounds either the centre of a long cylindrical coil, containing a bar of soft iron, or any point of an

annular coil, we can at a given moment suddenly establish or suppress a current of known strength I in the magnetising coil. The expression $\frac{m\mathrm{R}}{n'}$, which in the induced current corresponds to the make or break of the principal current, represents the total flow LI of magnetic induction which traverses one of the turns.

We could then determine experimentally the value of L, and from it deduce the coefficient of magnetisation k, particularly by formulæ (54) and (57). This is the principle of the method recently employed by Prof. Rowland.

560. WEBER'S HYPOTHESIS ON MAGNETISM AND DIAMAGNETISM. —We have seen above how Ampère explains magnetism by molecular currents; we may now examine the physical properties of these currents.

Consider one of the currents defined by the values L, I, and R, and let Q be the flow of external force in its contour; as the electromotive force is null, equation (10) of (518) becomes

$$\frac{d}{dt}(\mathrm{LI}+\mathrm{Q})+\mathrm{RI}=0.$$

We must also assume that the resistance is zero; it follows that

$$\mathrm{LI}+\mathrm{Q}=\mathrm{const}=\mathrm{LI}_0. \tag{64}$$

The strength I_0 is that of the current which would traverse the circuit in question, if the external flow of force were zero. If we suppose $\mathrm{I}_0=0$ (that is, the molecular current originally null), which would correspond to the case of a magnetic medium in the neutral state, we have finally

$$\mathrm{LI}=-\mathrm{Q}.$$

The current induced in the molecule by an external field produces therefore a flow of force contrary in sign to Q. In other words, the magnetisation equivalent to the current is of opposite sign to the magnetising force; the magnetisation of magnetic bodies cannot therefore be explained solely by currents induced in the molecules of the medium.

561. Such currents can, on the contrary, account for diamagnetic phenomena. Weber's hypothesis assumes that in each molecule of a diamagnetic medium there are channels along which currents may circulate without resistance. If these channels were in all directions, the molecule would be a perfect conductor. With a linear current

which was originally zero, the strength is given by equation (64). If θ is the angle which the magnetising force X makes with the perpendicular to the plane of the circuit, we have

$$Q = XA \cos\theta.$$

The magnetic moment of the current is IA, and its projection in the direction of the magnetising force is

$$IA \cos\theta = -\frac{XA^2}{L}\cos^2\theta.$$

Suppose that there are n molecules in unit volume, and that the axes of the circuits are distributed indifferently in all directions. The zone corresponding to the angle $d\theta$ about the direction of the magnetising force is $2\pi \sin\theta d\theta$, so that the mean value of $\cos^2\theta$ is

$$\frac{1}{4\pi}\int_0^\pi 2\pi \cos^2\theta \sin\theta \, d\theta = \frac{1}{3}.$$

The magnetic moment is therefore $-\frac{nA^2}{3L}X$; the magnetisation is directly opposed to the magnetising force, which is in conformity with the phenomena of diamagnetism, and the value of the coefficient of magnetisation is

$$k = -\frac{nA^2}{3L}.$$

If the distribution of the axes of the molecular channels is not uniform, the sum $\Sigma\frac{A^2}{L}\cos^2\theta$ extended to the whole of the molecules will have different values according to the direction of the magnetising force, and we thus come upon the known properties of anisotropic diamagnetic substances.

562. Suppose that each molecule is a perfect conductor, or (what amounts to the same thing) that it is surrounded by a layer, the conductivity of which is perfect.

The total flow of magnetic induction $LI + Q$ which traverses any circuit traced on the surface is constant.

It follows that the perpendicular component of the force at each point of the surface is constant. If the flow of induction which proceeds from the molecule is originally null, any external magnetic system will produce induced currents, such that the resultant

magnetic induction will be null on the whole surface, and in the interior of the layer which surrounds the molecule.

If we assume that the molecule has the form of a sphere of radius r, the currents on the surface produce in the interior a force $-X$, equal and of opposite sign to the external magnetic force; the sphere is magnetised uniformly with an intensity (355) equal to $-\frac{3X}{8\pi}$, and the value of its magnetic moment is $-\frac{1}{2}r^3X$.

If we assume that in the unit volume there are n equal spheres, so small and so far apart that they do not act on one another, the mean strength of the magnetisation of the medium will be $-\frac{n}{2}r^3X = -\frac{3hX}{8\pi}$, h being the ratio of the sum of the volumes of the small spheres to the total volume of the space which contains them.

563. Absolute Conducting Screens.—It is clear that these conclusions from formula (63) would apply also to a surface of finite extent which possessed absolute conductivity; the induced currents which any variation of the field would produce in this surface, would always be such that the flow relative to each portion of the surface would be constant—in other words, that the perpendicular component of the magnetic induction at each point would retain a fixed value. If then this component part was null at a given moment, it would remain null whatever were the variations of the field. It follows from this that a closed or unlimited surface, of resistance zero, is a complete screen for all points in the interior against the effects of the variation of the field on the other side of the surface; these effects reduce to the production of surface currents, which keep the field in the interior constant at zero.

The distribution of magnetic forces about a perfect conductor is quite comparable to the distribution of velocities in an incompressible fluid which surrounded the same bodies.

564. In order to explain magnetic phenomena on these considerations, it must be assumed that the primitive current in a molecule, or about a conducting channel, is not null, and we take the general equation (64). Considering always the case of a circuit, the perpendicular of which makes an angle θ with the direction of the magnetising force, we shall have

$$LI + XA\cos\theta = LI_0,$$

or

$$I = I_0 - \frac{XA}{L}\cos\theta.$$

If the primitive current I_0 was null, the moment of the action of the field on the molecule will be

$$-IAX \sin\theta = \frac{X^2A^2}{L} \sin\theta \cos\theta = \frac{X^2A^2}{2L} \sin 2\theta.$$

The equilibrium will be stable, for $\theta = \frac{\pi}{2}$, that is when the plane of the current is perpendicular to the direction of the field. This is the case, for instance, of a ring which is suddenly brought into a very powerful field.

In the general case, the moment of the couple which acts on the molecule may be written

$$\frac{X^2A^2}{L} \cos\theta \sin\theta - I_0AX \sin\theta = mX \sin\theta [BX \cos\theta - 1],$$

putting $m = I_0A$ and $B = \frac{A}{LI_0}$.

If we assume with Weber (427), that the reaction D of the medium is constant in magnitude and in direction, we shall have the equation of equilibrium,

$$X \sin\theta(1 - BX \cos\theta) = D \sin(\alpha - \theta).$$

The component, parallel to the magnetising force X, of the magnetic moment of the molecule is

$$IA \cos\theta = \left(I_0 - \frac{XA}{L} \cos\theta\right) A \cos\theta = m \cos\theta(1 - BX \cos\theta).$$

If the coefficient B is very small—that is, if the primitive molecular current I_0 is very powerful—we arrive again at Weber's formula for the magnetisation of magnetic substances. If this coefficient B is very great we obtain the phenomena of diamagnetism.

In intermediate cases, magnetisation will first of all be proportional to the magnetising force for small forces, it will then pass through a maximum, and afterwards decrease. Experiment does not seem to show any phenomenon of this kind, so that the occurrence of currents of molecular induction cannot be considered proved.

CHAPTER VI.

PROPERTIES OF THE ELECTROMAGNETIC FIELD.

565. MAXWELL'S THEORY.—The preceding considerations are sufficient, as we have seen, to account for all the phenomena of induction in linear conductors; but it is useful to regard the problem from another point of view, in which the influence of the medium is brought out, as in electrostatics. We shall explain here the principles of Maxwell's theory.

566. EQUATIONS OF THE MAGNETIC FIELD. — Suppose, for greater generality, that the conductors are situate in a magnetic medium whose coefficient of permeability (383) is equal to μ. When the total flow of magnetic induction Q through a closed circuit is annulled, it becomes the seat of a total electromotive force which puts in motion a quantity of electricity equal to $\frac{Q}{R}$.

This total electromotive force may be regarded as the resultant of the elementary electromotive forces acting on each of the elements of the circuit, and arising from the condition of the medium. At each point the elastic reaction of the medium, due to the suppression of the forces, has a determinate direction; it would produce on an element of the conductor ds, situate in the same direction, an electromotive force proportional to the length of this element, and which may be represented by Jds; the electromotive force on an element which made an angle ϵ with the direction, would be equal to $Jds\cos\epsilon$.

If the medium is homogeneous and isotropic, the electromotive force J for unit length is a function of the co-ordinates; it may be replaced by its components F, G, and H, parallel to the axes, and which would produce the electromotive forces Fdx, Gdy, and Hdz

on the projections dx, dy, and dz, of the element ds. We shall have then the equation

$$(1) \qquad Q = \int J ds \cos\epsilon = \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} \right) ds,$$

the integral being extended to the whole circuit.

On the other hand, if we consider any given surface S bounded by a circuit, and we denote by X, Y, and Z the components of the magnetic force at a point of the surface, and by α, β, and γ the cosines of the angles of this force with the perpendicular, the total flow of magnetic induction across the circuit is also expressed by the integral $\mu \int (X\alpha + Y\beta + Z\gamma)\, dS$, extended to the whole surface.

567. In order to determine the relations between the electromotive force J and the magnetic induction, we shall calculate the value of the integral $\int J ds \cos\epsilon$ for an infinitely small rectangular circuit with its centre at the origin of the co-ordinates, perpendicular to the x axis, and the sides of which are equal to dy and dz.

Let F_0, G_0, H_0 be the values of the functions F, G, H at the origin; we shall have, for the top side of the rectangle,

$$G_1 = G_0 + \frac{1}{2}\frac{\partial G}{\partial z} dz;$$

and for the bottom side

$$G_2 = G_0 - \frac{1}{2}\frac{\partial G}{\partial z} dz.$$

If we follow the contour in the direction in which the current would be produced by the suppression of the field, the sum of the two terms of the electromotive force corresponding to the sides parallel to the y axis, is $G_1 dy - G_2 dy$; we get thus

$$G_1 dy - G_2 dy = \frac{\partial G}{\partial z} dz dy.$$

We shall have, in like manner, for the two other sides,

$$-H_1 dz + H_2 dz = -\frac{\partial H}{\partial y} dy dz.$$

The electromotive force corresponding to the contour is then

$$\int J ds \cos\epsilon = \left(\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y}\right) dydz.$$

On the other hand, this expression should also represent the flow of magnetic induction through the circuit $dydz$ from the negative face; as the expression for this flow is $\mu X dydz$, the component μX is equal to $\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y}$.

The other components of magnetic induction parallel to the y axis and the x axis, will satisfy analogous conditions, which is given by the equation

$$(2) \qquad \begin{aligned} \mu X &= \frac{\partial G}{\partial z} - \frac{\partial H}{\partial y}, \\ \mu Y &= \frac{\partial H}{\partial x} - \frac{\partial F}{\partial z}, \\ \mu Z &= \frac{\partial F}{\partial y} - \frac{\partial G}{\partial x}. \end{aligned}$$

Let us now consider any given closed circuit; we may divide the surface into elements by two series of arbitrary curves. If we make the summation $\int J ds \cos\epsilon$ of the electromotive forces along the contour of all the elements, we shall obtain the total flow of magnetic induction across the circuit, and this sum will reduce to the single terms furnished by the primitive contour, for the sum of the portions of the curve common to two contiguous elements is null. The total electromotive force of a circuit, only depends then on the values of the components F, G, and H at the different points of the contour, and these components are connected with the magnetic induction by the equations (2).

568. We may observe that the component F represents at a given point, and for unit length parallel to the x axis, the total electromotive force which corresponds to the suppression of the field. This electromotive force may also be regarded as the integral, in respect of time, of the elementary electromotive forces which correspond to the gradual suppression of the field. The

product of the electromotive force P, which acts at a given time on unit length parallel to the axis, by the time dt, is equal to the corresponding diminution of the value of F, which gives

$$Pdt = -\frac{\partial F}{\partial t}dt,$$

or

$$P = -\frac{\partial F}{\partial t}.$$

In like manner, if Q and R are the analogous components along the other axes, we shall have, for the determination of the electromotive force at each instant, the three equations

$$(3) \qquad \begin{aligned} P &= -\frac{\partial F}{\partial t}, \\ Q &= -\frac{\partial G}{\partial t}, \\ R &= -\frac{\partial H}{\partial t}. \end{aligned}$$

The function $-P$ expresses the difference of potential, which is produced at the point in question between the two ends of a length equal to unity, parallel to the x axis, in consequence of the variations which at the same time take place in the value of the flow of magnetic induction.

If there were also electrical masses producing the potential ψ at the same point, the total difference of potential dx will be equal to $\frac{\partial F}{\partial t}dx - \frac{\partial \psi}{\partial x}dx$; but this latter part is an exact differential which itself disappears when the formula is applied to a closed circuit.

569. EQUATIONS OF CURRENTS.—Let u, v, w be the components of the current at a point—that is to say, the quantities of electricity which in unit time traverse unit surface, perpendicularly to each of the axes.

We know that the work of a current I on a unit pole, movable on a closed curve which traverses the plane of the current (452), is equal to $4\pi I$ for each complete turn of the pole about the current.

If we apply this property to the case of a pole which traverses a rectangular circuit $dydz$, the centre of which is at the origin of

the co-ordinates, that is to say, which enclose a surface across which the strength of the current is $u\,dy\,dz$, and if we determine the same work by the magnetic forces, we find by a calculation analogous to that applied to (567), the three new equations

$$(4)\qquad \begin{aligned} 4\pi u &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \\ 4\pi v &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \\ 4\pi w &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \end{aligned}$$

570. Potential Energy of Currents. — Let us consider various currents C_1, C_2, C_3, ... in which the strengths are I_1, I_2, I_3 ...; let L_1, L_2, L_3 be their coefficients of self-induction and $M_{1.2}$, $M_{1.3}$, $M_{2.3}$, ... their coefficients of mutual induction.

The flows of force which traverse each circuit are

$$\begin{aligned} Q_1 &= L_1 I_1 + M_{1.2} I_2 + M_{1.3} I_3 + \ldots\ldots, \\ Q_2 &= L_2 I_2 + M_{2.1} I_1 + M_{2.3} I_3 + \ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and the potential energy of the whole of the currents (524) is

$$(5)\qquad W = \frac{1}{2}(Q_1 I_1 + Q_2 I_2 + \ldots\ldots) = \frac{1}{2}\Sigma QI.$$

From equation (1), we have therefore

$$W = \frac{1}{2}\Sigma I \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} \right) ds.$$

Taking u, v, and w in the same signification as above (569), the components of the current across the section S of one of the conductors are

$$I\frac{dx}{ds} = uS,$$

$$I\frac{dy}{ds} = vS,$$

$$I\frac{dz}{ds} = wS.$$

If, further, we observe that the product $S ds$ represents an element of volume, we shall have

$$W = \frac{1}{2}\int\int\int (Fu + Gv + Hw)\, dxdydz.$$

Replacing finally the components u, v, w by their values taken from equations (4), we get

$$W = \frac{1}{8\pi}\int\int\int \left[F\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + G\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) + H\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)\right] dxdydz.$$

We may integrate by parts each of the terms, which give, for instance,

$$\int F \frac{\partial Y}{\partial z}\, dxdydz = FY dxdy - \int Y \frac{\partial F}{\partial z}\, dxdydz.$$

Repeating the same operation for all the other terms, we get

$$W = \frac{1}{8\pi}\int \Big[X(Hdz - Gdy)\, dx + Y(Fdx - Hdz)\, dy + \ldots\ldots \Big]$$
$$+ \frac{1}{8\pi}\int\int\int \left[X\left(\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y}\right) + Y\left(\frac{\partial H}{\partial x} - \frac{\partial F}{\partial z}\right) + Z\left(\frac{\partial F}{\partial y} - \frac{\partial G}{\partial x}\right)\right] dxdydz.$$

If we extend this expression to the whole space contained in a very distant surface, which comprises the whole system of the magnets and of the currents, the first integral relative to the surface itself is null, for the components of the magnetic force are inversely as the cube of the distance. On the other hand, replacing in the second integral the terms comprised within the brackets by their values drawn from equations (2), we get

$$W = \frac{\mu}{8\pi}\int\int\int (X^2 + Y^2 + Z^2)\, dxdydz,$$

or, if ϕ is the magnetic induction at a point—that is to say, the product of the force by μ,

$$W = \frac{1}{8\pi\mu}\int\int\int \phi^2\, dxdydz. \tag{6}$$

The potential energy of currents may therefore be expressed by an integral (5) which contains the currents themselves, or by an integral in respect of all parts of the space in which are magnetic forces. In the former case, the reciprocal action of the currents is considered as being directly exerted at a distance; in the second case, this action results from the elasticity of the intervening medium. If we adopt this point of view, we see that the potential energy of the medium at each point, for unit volume, is expressed by

$$W_1 = \frac{1}{8\pi\mu}\phi^2. \tag{7}$$

571. RELATIVE DISPLACEMENT OF CIRCUITS.—The formulæ (3) correspond to the case in which, the circuit being fixed, induction is due to the mean variation in the field; the quantities F, G, and H are at each point mere functions of the time. But if, while the field is variable, we displace or alter the configuration of the circuit, the co-ordinates x, y, z should themselves be considered as functions of the time. Equation (1) should then be written

$$Q = \int \left(F\frac{\partial x}{\partial s} + G\frac{\partial y}{\partial s} + H\frac{\partial z}{\partial s} \right) ds. \tag{8}$$

The electromotive force utilised in the time dt is $-dQ$, so that the electromotive force of induction e at a given moment is expressed by

$$\begin{aligned} e = -\frac{dQ}{dt} = &- \int \left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial x}\frac{\partial y}{\partial s} + \frac{\partial H}{\partial x}\frac{\partial z}{\partial s} \right) \frac{\partial x}{\partial t} ds \\ &- \int \left(\frac{\partial F}{\partial y}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial H}{\partial y}\frac{\partial z}{\partial s} \right) \frac{\partial y}{\partial t} ds \\ &- \int \left(\frac{\partial F}{\partial z}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial z}\frac{\partial y}{\partial s} + \frac{\partial H}{\partial z}\frac{\partial z}{\partial s} \right) \frac{\partial z}{\partial t} ds \\ &- \int \left(F\frac{\partial^2 x}{\partial s \partial t} + G\frac{\partial^2 y}{\partial s \partial t} + H\frac{\partial^2 z}{\partial s \partial t} \right) ds \\ &- \int \left(\frac{\partial F}{\partial t}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial H}{\partial t}\frac{\partial z}{\partial s} \right) ds. \end{aligned} \tag{9}$$

Consider the first term of the integral, and replace the partial differentials $\frac{\partial G}{\partial x}$ and $\frac{\partial H}{\partial x}$ by their values taken from equations (2); this term becomes

$$\int\left(\mu Y\frac{\partial z}{\partial s}-\mu Z\frac{\partial y}{\partial s}+\frac{\partial F}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial s}\right)\frac{\partial x}{\partial t}\,ds,$$

or

$$\int\left(\mu Y\frac{\partial z}{\partial s}-\mu Z\frac{\partial y}{\partial s}+\frac{\partial F}{\partial s}\right)\frac{\partial x}{\partial t}\,ds.$$

Analogous results will be obtained by treating the second and third terms in the same way. We may, moreover, observe that we have

$$\int\left(\frac{\partial F}{\partial s}\frac{\partial x}{\partial t}+F\frac{\partial^2 x}{\partial s\partial t}\right)ds=F\frac{\partial x}{\partial t},$$

and that the groups of this form disappear when the integral is extended to a closed circuit.

The value of the electromotive force reduces then to

$$e=\int\left[\left(\mu Y\frac{\partial z}{\partial t}-\mu Z\frac{\partial y}{\partial t}-\frac{\partial F}{\partial t}\right)\frac{\partial x}{\partial s}+\left(\mu Z\frac{\partial x}{\partial t}-\mu X\frac{\partial z}{\partial t}-\frac{\partial G}{\partial t}\right)\frac{\partial y}{\partial s}+..\right]ds.$$

Each of the three groups contained in the parenthesis represents the electromotive force which, at a given instant, acts on unit length parallel to one of the axes. If, further, we suppose that the field contains electrical masses giving a potential ψ, the most general values of the components P', Q', R' of the electromotive force will be

$$(10)\qquad \begin{aligned} P' &= \mu Y\frac{\partial z}{\partial t}-\mu Z\frac{\partial y}{\partial t}-\frac{\partial F}{\partial t}-\frac{\partial \psi}{\partial x},\\ Q' &= \mu Z\frac{\partial x}{\partial t}-\mu X\frac{\partial z}{\partial t}-\frac{\partial G}{\partial t}-\frac{\partial \psi}{\partial t},\\ R' &= \mu X\frac{\partial y}{\partial t}-\mu Y\frac{\partial x}{\partial t}-\frac{\partial H}{\partial t}-\frac{\partial \psi}{\partial z}. \end{aligned}$$

572. EQUATIONS OF THE ELECTRICAL FIELD.—Let us consider, lastly, an isotropical field in which continuous currents may coexist with currents corresponding to a variation in the electrical displacement (126).

If f, g, h are the components of the electrical displacement, the true quantities of electricity u', v', w', which in unit time traverse unit surface perpendicularly to the axis, consist of continuous currents u, v, w (569), and of variations of electrical displacement. We have then

$$\begin{aligned} u' &= u + \frac{\partial f}{\partial t}, \\ v' &= v + \frac{\partial g}{\partial t}, \\ w' &= w + \frac{\partial h}{\partial t}. \end{aligned} \tag{11}$$

The equations of the currents (4) become

$$\begin{aligned} 4\pi u' &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \\ 4\pi v' &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \\ 4\pi w' &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \end{aligned} \tag{12}$$

When the medium is a dielectric, it establishes equilibrium between the electromotive force, whose components are given by equation (10), and the elastic reactions developed by the displacement. If $\frac{4\pi}{K}$ is the value in electromagnetic units of the coefficient of electrical elasticity—that is to say, the ratio of the electromotive force to the displacement—we shall have

$$\begin{aligned} KP &= 4\pi f, \\ KQ &= 4\pi g, \\ KR &= 4\pi h. \end{aligned} \tag{13}$$

In the case of a conductor, on the contrary, the displacement is null, and if c is the specific conductivity of the medium, Ohm's law becomes

$$\begin{aligned} Pc &= u, \\ Qc &= v, \\ Rc &= w. \end{aligned} \tag{14}$$

Finally, if ρ is the density of the free electricity at the point in question, we shall have, with a dielectric, the condition

$$\rho + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0, \tag{15}$$

and, in the case of a conductor,

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{16}$$

The twenty values F, G, H; P, Q, R; X, Y, Z; u, v, w; f, g, h; u', v', w'; ρ and ψ can be determined from the twenty equations in numbers (2), (10), (11), (12), (13), (14), (15), and (16), when the conditions of the problem in each special case are known.

CHAPTER VII.

PHENOMENA OF INDUCTION IN NON-LINEAR CONDUCTORS.

573. MAGNETISM OF ROTATION.—Following up an observation by Gambey on the deadening of the oscillations of a compass needle, Arago showed, in 1824, that a magnetised needle placed above a disc which is in rapid rotation, is carried along by the disc, and tends to acquire a rotation in the same direction.

The action exerted on a pole in these three conditions has three components; the one tangential which impels the pole in the direction of the rotation, another perpendicular which tends to remove it from the disc, and finally a third directed along the radius. This latter is null when the pole is at a distance from the axis equal to about two-thirds of the radius of the disc; nearer the axis the pole is attracted to the centre; when nearer the edges, it seems repelled towards the edges.

The motion of the needle is more marked with a good conductor like copper, than with a metal which does not conduct so well, such as brass, and particularly antimony. When there are breaks in the continuity of the disc (as for instance with a radial saw cut) the effect produced is enfeebled.

These phenomena were at first ascribed to a special form of magnetisation, and were known as *magnetism of rotation.* They are really produced by induction currents developed in the metal; but it was only after Faraday's great discovery that they were ascribed to the true cause.

574. CONDUCTING SHELLS.—The problem raised by Arago's experiment is that of induction in a conductor of two dimensions. Maxwell solved this problem in a very elegant manner, by the use of a method analogous to that of electrical images.

Let us consider an infinitely thin homogeneous conductor, which may be supposed reduced to a surface, and in which exist, for any reason, electric currents which are not brought from without by electrodes—these currents are necessarily closed, and the stream-lines cannot intersect each other. The annular space comprised between two infinitely near currents, may be regarded as a linear circuit traversed by a current of strength $d\Phi$. This current may be replaced by a magnetic shell of the same strength and the same contour.

If the surface of the conductor is thus cut into infinitely thin bands by the stream-lines, it is seen that for an external point, the sum of the currents will be equivalent to a complex shell (333), the magnetic strength Φ of which is equal at each point to the sum of that of the superposed shells. On the positive face of the shell the currents are in the opposite direction to the hands of a watch, about the spaces where the value of Φ is a maximum. This value is null at the edge if the plate is bounded.

Along a current line the value of Φ is constant; the current lines are equipotential lines of the function Φ. An element dn of a line n, at right angles to the current line, is cut perpendicularly by a current of strength $\frac{d\Phi}{dn}dn$, in the direction of the right of an observer, who, along this line, would move towards points where the function Φ increases. Finally an element ds' of any given curve is cut by a current of strength $\frac{\partial\Phi}{\partial s'}ds'$, but which is no longer perpendicular.

The expression for the magnetic potential on the outside is

$$V = \int \Phi d\omega.$$

This function is discontinuous on traversing the surface; the two values V_2 and V_1 which it assumes on either side of the shell, on the positive and on the negative face (330) are connected by the equation,

$$V_2 - V_1 = 4\pi\Phi.$$

The component of the magnetic force perpendicular to the surface is continuous, for it represents, on either side, the flow of induction for unit surface. This is the case also with the tangential component along a stream-line, for then we have $\frac{\partial\Phi}{\partial s} = 0$.

On the contrary, the tangential component along a perpendicular to the stream-lines is discontinuous, and on both sides of the surface we have

$$\frac{\partial V_2}{\partial n} = \frac{\partial V_1}{\partial n} + 4\pi \frac{d\Phi}{dn}.$$

The value of V may also be expressed (360) as a function of the potential Q of a layer which covered the same surface, and the density of which is equal at each point to the strength Φ of the shell.

575. Case of a Plane Shell.—Let us consider, as a special case, a plane conducting plate in the plane xy, and suppose that the positive face of the currents is at the top; the potential of the corresponding magnetic shell at a point whose co-ordinates are x, y, and z, is expressed by (362)

$$(1) \qquad V = -\frac{\partial Q}{\partial z}.$$

The function Q which represents the potential of a layer whose density at each point is equal to the magnetic strength of the shell, is symmetrical in respect of the xy plane, and does not change when z is replaced by $-z$. The function V, on the other hand, changes its sign with z, and its absolute value is the same at two points which are symmetrical in reference to the shell. We have therefore, for corresponding points of the positive and of the negative face,

$$(2) \qquad \begin{aligned} V_2 &= -\frac{\partial Q}{\partial z} = 2\pi\Phi, \\ V_1 &= -\frac{\partial Q}{\partial z} = -2\pi\Phi. \end{aligned}$$

The components X and Y of the magnetic force parallel to the axes on the positive face, and the values of these components X′ and Y′ on the negative face, are given by the equations

$$(3) \qquad \begin{aligned} X &= -\frac{\partial V_2}{\partial x} = -2\pi\frac{\partial \Phi}{\partial x} = -X', \\ Y &= -\frac{\partial V_2}{\partial y} = -2\pi\frac{\partial \Phi}{\partial y} = -Y'. \end{aligned}$$

576. Let us now investigate the currents in the plate. The component u of the current parallel to the x axis, which intercepts unit length parallel to the y axis, is

$$u = -\frac{\partial \Phi}{\partial y},$$

and the component v of the current parallel to the y axis

$$v = \frac{\partial \Phi}{\partial x}.$$

If σ be the resistance of the plate for unit surface, the fall of electrical potential for unit length will be σu parallel to the x axis, and σv parallel to the y axis; this fall of potential is but the electromotive force in the same directions. Hence, from the expressions found above (568), we have for a field variable with the time,

$$-\sigma \frac{\partial \Phi}{\partial y} = -\frac{\partial F}{\partial t},$$
$$\sigma \frac{\partial \Phi}{\partial x} = -\frac{\partial G}{\partial t}.$$

The equations (2) give, for the positive face,

$$\Phi = -\frac{1}{2\pi}\frac{\partial Q}{\partial z};$$

from this follows

$$\frac{\sigma}{2\pi}\frac{\partial^2 Q}{\partial y \partial z} = -\frac{\partial F}{\partial t},$$
(4)
$$-\frac{\sigma}{2\pi}\frac{\partial^2 Q}{\partial z \partial x} = -\frac{\partial G}{\partial t}.$$

The general equations (567) which connect the components of the total electromotive force of induction with the magnetic potential give, in the present case,

$$\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y} = X = -\frac{\partial V}{\partial x} = \frac{\partial^2 Q}{\partial x \partial z},$$
$$\frac{\partial H}{\partial x} - \frac{\partial F}{\partial z} = Y = -\frac{\partial V}{\partial y} = \frac{\partial^2 Q}{\partial y \partial z},$$
$$\frac{\partial F}{\partial y} - \frac{\partial G}{\partial x} = Z = -\frac{\partial V}{\partial z} = \frac{\partial^2 Q}{\partial z^2}.$$

These equations are satisfied by putting

$$F = -\frac{\partial Q}{\partial y}, \qquad G = \frac{\partial Q}{\partial x}, \qquad H = 0.$$

577. If, at the same time, there is an external system of magnets or of currents, we know (498) that its action on a surface surrounding it is equivalent to that of a suitable system of surface currents. The magnetic potential of this system, in the positive surface of the shell under consideration, might then be expressed by a function Q′, analogous to the function Q; we shall have then

$$V = -\frac{\partial (Q + Q')}{\partial z},$$

which gives

$$F = -\frac{\partial (Q + Q')}{\partial y}, \qquad G = \frac{\partial (Q + Q')}{\partial x}.$$

Putting $\frac{\sigma}{2\pi} = R$, equations (4) become

$$(5) \qquad \begin{aligned} R\frac{\partial^2 Q}{\partial y \partial z} &= \frac{\partial^2 (Q + Q')}{\partial y \partial t}, \\ R\frac{\partial^2 Q}{\partial z \partial x} &= \frac{\partial^2 (Q + Q')}{\partial x \partial t}. \end{aligned}$$

Integrating the first with respect to y, or the second with respect to x, we get

$$(6) \qquad R\frac{\partial Q}{\partial z} = \frac{\partial (Q + Q')}{\partial t}.$$

For a complete integration we should add an arbitrary function of t, but it must be observed that this function will disappear whenever we take a partial differential in respect of x or y to calculate the components of the current; it need not then be taken into account.

578. Let us suppose, at first, that there is no external magnetic body—that is to say that Q′ is null. This would be the case of a system of currents set up in the shell and left to themselves; these currents would act upon each other by their mutual induction, and

would rapidly lose their energy owing to the resistance of the conductor. Equation

$$R\frac{\partial Q}{\partial z}=\frac{\partial Q}{\partial t}$$

gives then

$$Q=f(x, y, z+Rt). \tag{7}$$

Hence the value of the function Q, at the time t, in a point at a distance z from the plane on the side of the positive face, and the co-ordinates of which are x, y, z, is the same as for the time $t=0$, at the point x, y, and $z+Rt$.

It follows that if a system of currents has been established in an unlimited and uniform plane, and then left to itself, the magnetic effect of these currents on a point on the side of the positive face, is the same as if the plane moved parallel to itself along the normal and on the negative side with a constant velocity R. The diminution of electromotive force in consequence of the enfeeblement of the currents is exactly represented by the diminution of the magnetic field which results at each point from this imaginary motion.

579. MAGNETIC IMAGES.—The integral of equation (6), in respect of t, gives for points on the surface of the shell

$$Q+Q'=\int R\frac{\partial Q}{\partial z}\,dt. \tag{8}$$

If we suppose that, Q and Q′ being at first null, the external system is suddenly created on the positive side, in such a manner that the corresponding potential Q′ passes suddenly from 0 to Q′, we shall have at the outset and for the surface of the shell, since the integral is null,

$$Q=-Q'.$$

Hence, for all points of the plate, and therefore for all points of the negative face, the initial system of currents produces an effect equal, and of opposite sign, to that of the real system placed on the positive side. Their effect is then the same as that of a magnetic system identical and of contrary sign to the real system, and which would coincide with it.

o o

For points on the positive side, the effect of the currents is the same as that of a system of the same sign as the real systems, and which would be symmetrical with it in respect of the conducting plane; we shall call it the *positive image* of the system.

The action of the currents of each side of the shell may then be considered as produced by an *image* of the magnetic system, positive or negative—that is to say, of the same sign as the system, or of the contrary sign, according as the point in question is on the positive or negative side of the shell.

If the conductivity of the shell were infinite, we should have $R = 0$; the second member of the equation (8) will always be zero, and the condition $Q = -Q'$ will be always satisfied. The plate will be an absolute screen (563) for all points on the negative side. The currents will be permanent, and their effect will be represented at each instant, for all points of space, by that of one or the other of the two fixed images.

In a real sheet the resistance R has a finite value. The currents produced by the sudden introduction of a magnetic system begin at once to decrease, and their effect on each side is at each instant represented by that of two images of the system, which would recede perpendicularly from the sheet on each side with the velocity R.

580. INDUCTION OF A MOVABLE MAGNETIC SYSTEM.—The principle of images enables us to determine induced currents by the variations of any given magnetic system on the positive side of the shell.

The function Q', which determines the magnetic action, will vary by $\frac{\partial Q'}{\partial t}dt$, while the system itself will vary by $\frac{\partial M}{\partial t}dt$. We may consider this latter system as being itself a magnetic system, and suppose that at the instant t there is suddenly formed on the negative side of the sheet a positive image of $\frac{\partial M}{\partial t}dt$, which then moves perpendicularly away with a constant velocity R. If the system varies continuously, we may suppose that the different images of the variations, relative to the different intervals of time, move according to the same law as soon as they are formed, and thus form continuous trails of images.

581. Suppose, for instance, that a positive pole $+m$ moves in a right line with a constant velocity u, parallel to the shell, and let us assume that this pole has been suddenly created at the point A (Fig. 120), which gives rise to an image $+m$ at the symmetrical

point B. After an infinitely small time δt, the pole comes to A′ (Fig. 121), at the distance $u\delta t$; it is as if we suddenly and simultaneously brought a pole $-m$ to A, and a pole $+m$ to A′, producing images of the same sign at B and B′; but at this moment the first image $+m$, which was at B, has come to C at the distance $R\delta t$. At the period $2\delta t$, the mass $+m$ is at A″ (Fig. 122). There are then two images equal to $-m$ at C and B′, and three positive images at B″, C′ and D; and so on.

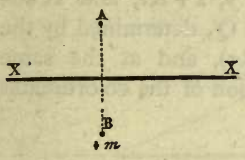

Fig. 120.

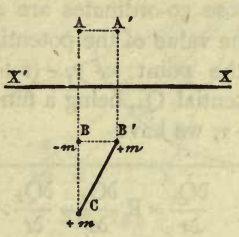

Fig. 121.

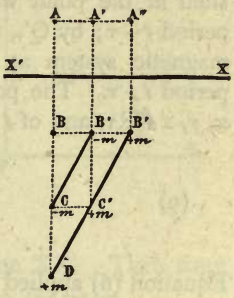

Fig. 122.

If the motion is continuous, it will be seen that the action which the movable pole undergoes is that of two magnetic lines, one positive and the other negative. If U is the resultant $\sqrt{u^2+R^2}$ of the two velocities, the density of these two lines will be $\frac{m}{U\delta t}$ and their distance $\frac{Ru}{U}\delta t$; the product of the density by the distance is then $m\frac{Ru}{U^2}$. These two lines form an unlimited magnetised ribbon, which starts from the point symmetrical with the actual position of the pole, is situate in a plane passing through the trajectory of the pole, and makes with the plane of the sheet an angle whose tangent is equal to $\frac{R}{u}$.

582. If the pole $+m$, instead of describing a right line, rotates uniformly about an axis perpendicular to the sheet, we may suppose that the band which precedes, forms a helix on the right cylinder which has the circumference described by this pole for its base.

For a magnet reduced to its two poles, and turning about its centre, the induced currents are also equivalent to the system of two magnetic ribbons coiled on the same cylinder. Lastly, the currents produced by the displacement of any magnetic system, rotating uniformly, are equivalent to a system of magnetic ribbons which correspond, point to point, to the different masses of the system.

583. CALCULATION OF THE ACTION OF INDUCED CURRENTS.—To calculate the effect of these images, let us denote by Q_τ the value of the potential Q, determined by the currents of the shell at the point whose co-ordinates are x, y, $z+R\tau$, and at the period $t-\tau$; by Q'_τ, the value of the potential Q', determined by the magnetic system at the point x, y, $-(z+R\tau)$, and at the same period $t-\tau$. The potential Q_τ, being a function of the co-ordinates x, y, $z+R\tau$, and of $t-\tau$, we have

$$\frac{\partial Q_\tau}{\partial \tau} = R\frac{\partial Q_\tau}{\partial z} - \frac{\partial Q_\tau}{\partial t}. \tag{9}$$

Equation (6) applied to this function becomes then

$$\frac{\partial Q_\tau}{\partial \tau} = \frac{\partial Q'_\tau}{\partial t}.$$

Integrating this equation in reference to τ between the limits $\tau = 0$, and $\tau = \infty$, we shall have the value of the function Q for the period t, which gives

$$Q = \int_0^\infty \frac{\partial Q'_\tau}{\partial t}\, d\tau = \frac{\partial}{\partial t}\int_0^\infty Q'_\tau\, d\tau. \tag{10}$$

The function Q on which the solution of the problem depends, since it enables us to calculate the action of induced currents at each point, is thus determined by the function Q'_τ defined at each instant by the condition and the motion of the external magnetic system.

584. CASE OF A SINGLE POLE.—We may apply this method of calculation to the case, considered above, of a single pole of mass m, which moves uniformly in a rectilinear path in the presence of an unlimited conducting plane; but it is simpler to treat the problem directly by the consideration of magnetic images.

Let us first examine the action which is exerted on a point A (Fig. 123), by a homogeneous magnetic line of density λ, situate along the right line XX'. The action of an element MM' or ds, at the distance r from the point A, is equal to $\frac{\lambda ds}{r^2}$. Let us denote by da the angle between the two lines AM and AM', and let fall the two

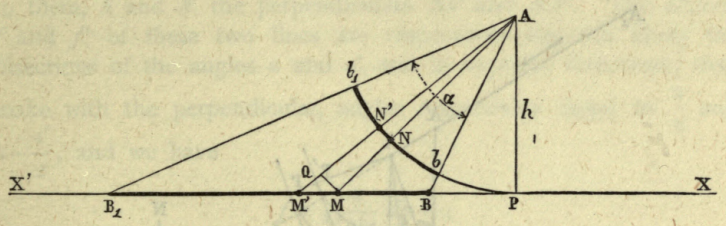

Fig. 123.

perpendiculars MQ upon AM' and AP, or h on XX'. Then from the similar triangles APM and MQM' we have the ratio

$$\frac{MQ}{MM'} = \frac{AP}{AM}, \quad \text{or} \quad \frac{rda}{ds} = \frac{h}{r};$$

from this follows

$$\frac{\lambda ds}{r^2} = \lambda \frac{da}{h} = \lambda \frac{hda}{h^2} = \lambda \frac{NN'}{h^2}.$$

The action of the element ds on the point A is therefore equal to that of the element NN' of the circumference of radius h, which has the same density λ, or to that of the corresponding element of the circumference of radius 1, where the density is $\frac{\lambda}{h}$. If the line in question was bounded at the points B and B_1, its action is equal to that of the arc of circle bb_1. We at once see that this latter action is proportional to the length of the chord, which gives

$$f = \frac{\lambda}{h^2} 2h \sin\frac{a}{2} = \frac{2\lambda}{h} \sin\frac{a}{2}.$$

585. Let us now suppose that a movable pole, of mass m, moving uniformly in a straight line with the velocity u, and having been in motion from an infinite time before the period in question, is at the point A, at a distance c from the conducting shell X'X (Fig. 124), and comes from a direction A_1A such that it has not yet traversed the shell. Let θ be the angle of this direction with the perpendicular N to the shell.

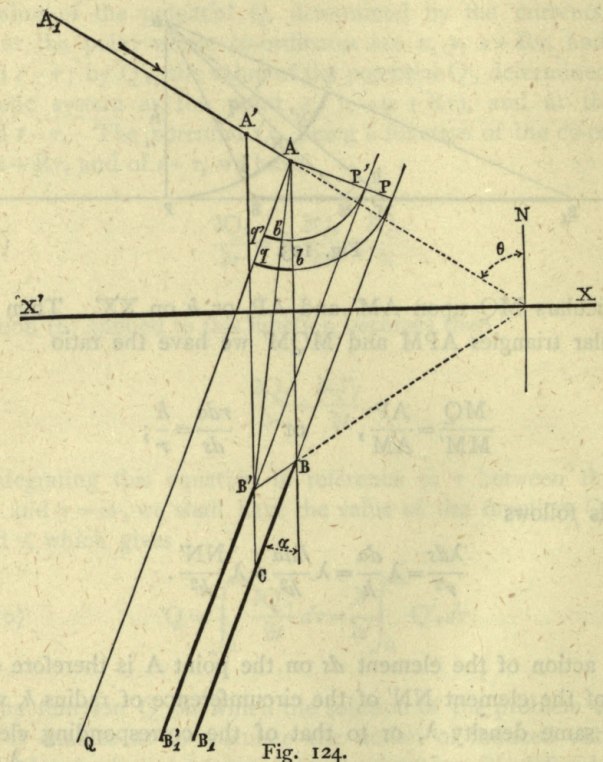

Fig. 124.

The angle α, which the magnetic trails B'_1B' and B_1B, relative to the two successive positions A' and A of the pole make with N, is defined by the triangle BB'C, which gives

$$U^2 = R^2 + u^2 + 2Ru\cos\theta,$$

$$\sin\alpha = \frac{u\sin\theta}{U}.$$

The distance of the two lines along B′B is $u\delta t$, and, along the common perpendicular, $R\delta t \sin \alpha$. Lastly, the density of each of them is

$$\lambda = \mp \frac{m}{U\delta t}.$$

Let us draw the right line AQ parallel to the magnetic trails; let α and α' be the angles of the arcs bq and $b'q'$ which correspond to them, h and h' the perpendiculars AP and A′P′. The actions f and f' of these two lines are respectively directed along the bisections of the angles α and α', and in opposite directions; they make with the perpendicular, angles respectively equal to $\frac{\alpha}{2}$ and $\alpha - \frac{\alpha'}{2}$, and we have

$$f = \frac{2m}{Uh\delta t} \sin \frac{\alpha}{2},$$

$$f' = \frac{2m}{Uh'\delta t} \sin \frac{\alpha'}{2}.$$

The components Z and X of the resultant force, measured perpendicularly, and parallel to the sheet, the first upwards, and the second in the direction of the motion of the movable pole, are

$$Z = f \cos \frac{\alpha}{2} - f' \cos \left(\alpha - \frac{\alpha'}{2}\right) = \frac{m}{U\delta t}\left[\frac{\sin \alpha}{h} - \frac{2 \sin \frac{\alpha'}{2} \cos \left(\alpha - \frac{\alpha'}{2}\right)}{h'}\right],$$

$$X = f \sin \frac{\alpha}{2} - f' \sin \left(\alpha - \frac{\alpha'}{2}\right) = \frac{2m}{U\delta t}\left[\frac{\sin^2 \frac{\alpha}{2}}{h} - \frac{\sin \frac{\alpha'}{2} \sin \left(\alpha - \frac{\alpha'}{2}\right)}{h'}\right].$$

We have further

$$h = 2c \sin \alpha,$$

$$h' = h - R \sin \alpha \, \delta t = h\left(1 - \frac{R \sin \alpha}{h} \delta t\right) = h\left(1 - \frac{R}{2c} \delta t\right),$$

or, disregarding the infinitely small quantities of the second order,

$$\frac{1}{h'} = \frac{1}{h}\left(1 + \frac{R}{2c} \delta t\right).$$

The triangle ABB', gives also

$$\frac{\alpha - \alpha'}{u\delta t} = \frac{\sin\theta}{2c},$$

or

$$\alpha' = \alpha - \frac{u\sin\theta}{2c}\delta t = \alpha - \frac{\mathrm{U}\sin\alpha}{2c}\delta t.$$

Substituting these values of h' and α' in the expressions of the components, and neglecting quantities of the second order, we get

$$\mathrm{Z} = \frac{m}{4c^2}\frac{\mathrm{U}-\mathrm{R}}{\mathrm{U}},$$

$$\mathrm{X} = -\frac{m}{4c^2}\frac{\mathrm{R}}{\mathrm{U}}\tan\frac{\alpha}{2}.$$

The force itself makes with the surface an angle β, determined by the conditions

$$\tan\beta = \frac{\mathrm{Z}}{-\mathrm{X}} = \frac{\mathrm{U}-\mathrm{R}}{\mathrm{R}}\cot\frac{\alpha}{2}.$$

The induced currents act thus in opposition to the motion of the pole, as might be foreseen; but it is not directly opposed.

If the motion of the pole is perpendicular to that of the shell, we have $\theta = 0$, $\alpha = 0$, and $\mathrm{U} = \mathrm{R} + u$; we get then

$$\mathrm{Z} = \frac{m}{4c^2}\frac{u}{\mathrm{R}+u},$$

$$\mathrm{X} = 0.$$

The action is the same as that of a single mass equal to $m\frac{u}{\mathrm{R}+u}$, situate at each instant at the point B symmetrical with the position of the pole, or of a mass $\frac{m}{4}\frac{u}{\mathrm{R}+u}$ placed at the base of the perpendicular let fall from the pole upon the plane.

If the pole is displaced parallel to the plane, we have

$$\mathrm{U}^2 = \mathrm{R}^2 + u^2, \qquad \tan\alpha = \frac{u}{\mathrm{R}};$$

which gives

$$Z = \frac{m}{4c^2}\frac{u^2}{U(R+U)}, \qquad F = \frac{mu}{4c^2(R+U)},$$

$$X = -\frac{m}{4c^2}\frac{uR}{U(R+U)}, \quad \tan\beta = \frac{u}{R} = \tan\alpha.$$

The force is perpendicular therefore to the direction of the magnetic trail; it is the same as if there were a mass equal to $\frac{m}{4}\frac{U^2}{(R+U)u}$ in the plane in its direction.

We may suppose again that the force F is produced by an infinitely small magnet situate in the plane. Applying Gauss' formula (154), for instance, we find that this magnet is situate behind the projection of the pole, at a distance x, defined by the equation

$$x = \frac{3cR}{4u}\left(\sqrt{1+\frac{8u^2}{9R^2}} - 1\right),$$

the magnetisation being parallel to the direction of the motion. The moment of this magnet would moreover be easy to calculate.

The component X is a maximum, for a given value of R, when $u = 1.27R$; it is zero for $R = 0$ and $R = \infty$.

The component Z tends to move the pole from the plate; it increases with the velocity, and tends towards the value $\frac{1}{4c^2}$, when the velocity tends towards infinity.

586. In the case of a uniform rectilinear motion parallel to the plane, we may consider the phenomena in still another manner.

The magnetic ribbon which starts from the point B (Fig. 125), consists of two magnetic lines, the density of which is $\frac{m}{U\delta t}$, and the horizontal distance $u\delta t$.

Let us denote by s the distance M'B from any point M', and by x' the distance M'K. For an element ds of the ribbon the magnetic moment ϖ' is

$$\varpi' = m\frac{u}{V}ds = mds\sin\alpha.$$

As $x' = s\sin\alpha$, we have $\varpi' = mdx'$.

The action on A of this infinitely small magnet placed at M′, at the distance r', is equivalent to that of an infinitely small magnet of moment ϖ placed at M, at the distance r, and such that

$$\frac{\varpi}{r^3} = \frac{\varpi'}{r'^3} = \frac{m dx'}{r'^3}.$$

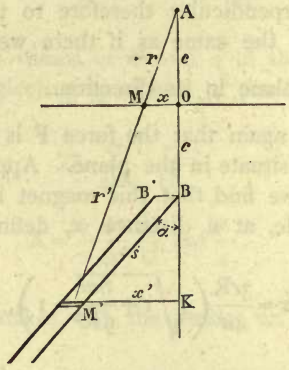

Fig. 125.

If x and c are the distances MO and AO, we have

$$\frac{r}{r'} = \frac{x}{x'} = \frac{c}{2c + x' \cot \alpha} = \frac{c - x \cot \alpha}{2c}.$$

From which we get

$$\varpi = \frac{m}{4} \frac{c - x \cot \alpha}{c} dx.$$

It is seen that the action of the induced currents on the point A is equivalent to that of a complex solenoid (327) situate on the plane starting from the point O, in a direction contrary to that of the motion, and the magnetic strength of which at each point will be

$$\frac{\varpi}{dx} = \frac{m}{4} \frac{c - x \cot \alpha}{c}.$$

We find accordingly, for the components X and Z of the force, the same values as before.

587. ARAGO'S EXPERIMENT.—This method may be generalised. Suppose that the pole describes a circle of radius a, in the opposite direction of the hands of a watch when looked at from above. The action of the induced currents is that of two homogeneous helices of contrary signs, or of a helicoidal ribbon BM′ (Fig. 126)

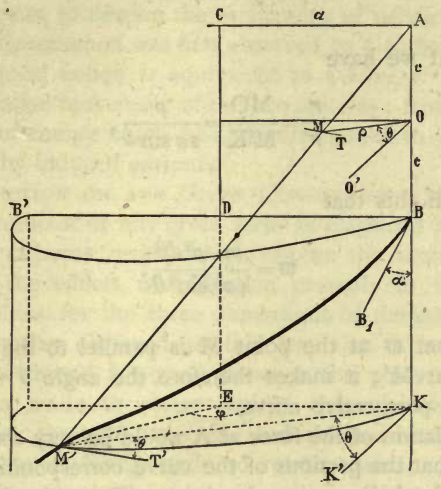

Fig. 126.

coiled on the cylinder, with the axis of rotation for its axis, and passing through the pole. Each element of this helix is magnetised along a tangent to the cylinder, drawn perpendicularly to the axis, and its action on the pole is equivalent to that of an infinitely small magnet situate at a point M in the conducting plane. The locus of the point M is the perspective curve of the helix seen from the point A. If r is the radius vector MO, θ the angle which it makes with the tangent at the point O, and observing that the angle θ is half the angle ϕ of the two planes passing through the axis, and through the points B and M′, we have, by the triangles AMO and AM′K,

$$\frac{\rho}{c} = \frac{\mathrm{M'K}}{\mathrm{AK}} = \frac{2a \sin \frac{\phi}{2}}{2c + a\phi \cot \alpha} = \frac{a \sin \theta}{c + a\theta \cot \alpha}.$$

This curve consists of a series of closed rings which have a common tangent at the point O.

The arc KM′ being equal to $a\phi$ or $2a\theta$, the magnetic moment ϖ of the element of the helix at M′ is $2mad\theta$. The magnetic moment ϖ of the corresponding magnet at M is given by the ratio

$$\frac{\varpi}{r^3}=\frac{\varpi'}{r'^3}=\frac{2ma}{r'^3}d\theta.$$

Observing that we have

$$\frac{r}{r'}=\frac{\mathrm{MO}}{\mathrm{M'K}}=\frac{\rho}{2a\sin\theta},$$

it follows from this that

$$\varpi=\frac{m}{4a^2}\frac{\rho^3 d\theta}{\sin^3\theta}.$$

The magnet ϖ at the point M is parallel to the magnetisation of the helix at M′; it makes therefore the angle θ with the radius vector of the perspective curve.

The calculation of the force at A would be very complicated, but it is evident that the portions of the curve corresponding to the first part BM′ of the helix are predominant. From the direction of the elementary magnets on the perspective curve, we see that the action on the point A will have a vertical component, another directly opposite to the velocity of the pole, and a third directed towards the centre of the circumference which it describes. The entire system is then equivalent to a small magnet placed behind the point O, perpendicular to a radius of the disc, which makes a certain angle from the opposite side of the motion with the radius corresponding to the pole, the magnetisation being in the direction of the motion if the pole in question is a north pole.

If the plane was unlimited, this magnet would be at a distance from the axis greater than that of the pole; but the action of the edges is to bring it more and more towards the centre in proportion as the radius diminishes. We thus find all the peculiar features of Arago's experiments, among others the fact that the radial component is centripetal so long as the pole is away from the edges, and that it becomes centrifugal as it approaches them.

588. If the pole describes any given curve parallel to the plane, we should obtain in the same way, by the trail of corresponding magnetic images, the magnetisation at each point of the perspective

curve. The action on the point A in the case of an unlimited plane, will still have a vertical component, another directly opposed to the motion, and a third perpendicular to the trajectory of the pole, and directed towards the concavity of the curve.

589. Damping of Compass Needles.—The action of a conducting disc on a magnetic system in motion is used in compasses, and galvanometers, to deaden the oscillations of needles, in the form in which the phenomenon was first observed by Gambey.

This reciprocal action is equivalent to a kind of friction which hinders the relative movement of the two systems; from this follows an absorption of energy which exactly corresponds to the heating of the conductor by induced currents.

590. Induction on any Given Conductor.—More generally, whenever a conductor of any given form is displaced in a magnetic field, induced currents result, which oppose the motion; but the calculation of the effects of induction presents in that case the greatest difficulties. for the three dimensions of the conductor come into play. Faraday observed in this way that if a copper cube suspended by a thread is placed between the poles of an electromagnet and is made to rotate rapidly, the cube will stop when a current passes through the coils, and a considerable resistance is experienced to its being made to rotate again.

Foucault conceived the idea of utilising this experiment to render evident the heating of a conductor. By means of a system of toothed wheels worked by a handle, he maintains a conducting disc in rotation between the poles of a very powerful electromagnet; the work expended is considerable, and the temperature of the disc rises very rapidly. The measurement of the work expended, and of the corresponding heat even furnishes a means of determining the mechanical equivalent of heat; this is the principle of the method used by M. Violle.

CHAPTER VIII.

OPTICAL PHENOMENA.

591. FARADAY'S DISCOVERY.—After prolonged researches, which for a long time were unfruitful, Faraday discovered in 1845 that a transparent body, though itself destitute of rotatory power, becomes capable, under the influence of magnetism, of rotating the plane of polarization of a luminous ray. The effect is at its maximum when the polarized ray traverses the body parallel to the lines of force; it is zero when the two directions are at right angles.

This phenomenon, which was first observed in the case of heavy flint glass, is produced in all single refracting liquids and solids; the action of magnetism is less perceptible in double refracting bodies; it is extremely feeble in gases and vapours, and it is only by quite recent experiments that it has been ascertained to exist. Bodies naturally endowed with rotatory power give rise to the same phenomenon; the two rotations become added or substracted according to their respective directions.

592. POSITIVE AND NEGATIVE BODIES.—All the bodies examined by Faraday rotate the plane of polarization in the same direction under the influence of magnetism; it is that direction of the current which, revolving around the ray, would give to the field its actual direction. All these substances are diamagnetic. Verdet found that most magnetic substances (for instance, solutions of ferric chloride in alcohol or ether) cause the plane of polarization to rotate in the opposite direction. If we consider the former rotation as positive, we may in general, though not with absolute strictness, assert that diamagnetic substances turn the plane of polarization in the positive direction, and magnetic substances in the negative direction.

593. There is an important difference in the way in which the rotation of the plane of polarization takes place, according as we consider the natural rotation or the magnetic rotation. In both

cases the angle of rotation is proportional, other things being equal, to the thickness of the medium traversed; but in quartz, in solution of sugar, in essence of turpentine, the rotation is connected with the propagation of light in such a manner that it is always in the same direction for the observer who receives the rays. It follows from this, that if the ray, after having traversed the transparent substance, returns on its own path after having been perpendicularly reflected, it undergoes a rotation which is equal and opposite to the first, and the plane of polarization reverts to its primitive position at its starting-point.

The magnetic rotation, on the contrary, is independent of the direction of the propagation, and only depends on the direction of the magnetic force. The ray, which returns on its own path after a normal reflection, undergoes a rotation in the same absolute direction, which adds on to the first; in this way, by causing the ray to be perpendicularly reflected an unequal number of times, $2n+1$, we may observe the same rotation as if it had traversed a layer of the substance $2n+1$ times the thickness.

594. VERDET'S LAWS.—Verdet proved, experimentally, that for a homogeneously polarized ray, the rotation of the plane of polarization is proportional:

1st. To the thickness traversed;

2nd. To the component of the magnetic force in the direction of the ray;

3rd. To a coefficient depending on the nature of the body, and which is positive or negative, according as the body is diamagnetic or magnetic.

These laws may be summarised in the following statement:

The rotation of the plane of polarization between two points is proportional to the difference of magnetic potential between these points.

Let V and V′ be the values of the potential at two points A and A′ in the path of the ray; the angle θ by which the plane of polarization is turned between these two points will be expressed by the ratio

$$\theta = \omega\,(V - V'),$$

ω being the rotation which for a given substance corresponds to a difference of potential equal to unity. This quantity is known as *Verdet's constant;* it defines the magnetic rotatory power of the body.

Verdet demonstrated that when a salt is dissolved in water the water and the salt each bring into the solution their special rotatory power; the rotation produced by the solution is the algebraical sum of the rotation due to each of the bodies composing it. Thus, water having a positive rotatory power, and ferric chloride a negative rotatory power, a solution of ferric chloride rotates the plane of polarization in one direction or the other, according to its concentration. The law may therefore be considered to be general.

595. Magnetic Rotatory Dispersion.—For the same body the value of the constant ω varies with the wave length. The direction of the variation is the same as for the natural rotation; in both cases, in fact, the rotation is approximately in the inverse ratio of the square of the wave length. In fact, the products of the rotation by the square of the wave length increases as the wave length diminishes. Both in the case of natural and of magnetic rotation, the substances for which the increase is most marked are just those with the greatest dispersive power.

Some months after the publication of Faraday's discoveries, Sir George Airy remarked that the phenomena could be accounted for, by adding to the known equations for the vibratory motion of isotropic substances, certain terms proportional to the differentials of the odd orders of the displacements in respect of the time. Among the various formulæ at which we arrive, by making special hypotheses as to the nature of the terms to be added to the equation, the following

$$\omega = m\frac{n^2}{\lambda^2}\left(n - \lambda\frac{dn}{d\lambda}\right),$$

in which m is a constant, and n the refractive index of the substance for a ray of wave length λ, gives an almost complete agreement with experiment. The constant m, as we shall afterwards see, will be inversely as the magnetic permeability.

M. H. Becquerel observed that the quotient of the rotatory power by the product $n^2(n^2-1)$ varies very little with different substances, and that this quotient is constant for bodies of the same chemical family.

596. Mr. Kerr's Experiment.—When a ray of polarized light is reflected from the pole of a magnet, its plane of polarization, from Kerr's experiments, experiences a manifest rotation; it is advantageous to make the reflection perpendicular in order to avoid the effects of elliptical polarization. On a positive or north pole,

the rotation is towards an observer's right—that is to say, inversely as the current which would produce the magnetisation. It is difficult to assert whether this is a simple magnetic rotation due to the gas, or, as Kerr believes, a new phenomenon.

597. EXPLANATION OF ROTATORY POLARIZATION. — The theoretical principles put forth by Fresnel to explain the rotatory polarization of quartz and of active liquids, may be applied to magnetic rotatory polarization.

It is known that a ray of light polarized in a plane is equivalent to two rays polarized circularly in opposite directions of the same period, moving with the same velocity, and the amplitude of whose vibration is half that of the resultant rectilinear vibration.

In order to get a conception of each of these circular rays, we shall assume that, all the molecules in the same straight line being disturbed from their position of equilibrium, and arranged along a helix having this right line as axis, a uniform rotation about the axis is imparted to the system. Each point of the system, which represents a vibrating molecule, describes a circumference about the axis, and at the same time the helix acquires an apparent longitudinal motion which represents the propagation of the undulation. The wave length, which is the distance traversed during a period, is represented by the thread of the screw. If the screw is right-handed, like an ordinary one, the vibration is in the direction of the hands of a watch for an observer towards whom the propagation takes place. The ray is said to be circularly polarized to the right, or more simply, that it is a right circular ray. The circular ray is left when the vibration is in the contrary direction to the hands of a watch for an observer towards whom it is moving.

If we superpose in this way two helices in opposite directions, starting from the same point A, and if each molecule of the medium shares this double motion, the successive positions which it will occupy in consequence of the two circular vibrations, will always be symmetrical in reference to the plane passing through the ray, and the point A; the resultant vibratory motion is always in this plane, and therefore the ray remains rectilinearly polarized in its original plane.

It may be assumed that matters take place in this way when a ray polarized rectilinearly traverses a transparent isotropic substance in the natural state, like Faraday's flint glass. But if this glass is placed in the magnetic field—for instance, inside a cylindrical coil—and if the light is propagated parallel to the lines of force,

and in that direction, the ray is still polarized on emergence, but the plane of polarization has turned through a certain angle in the direction of the external currents of the coil—that is, towards the left of an observer who receives the ray. The rotation is towards the right, on the contrary, if the light travels in the opposite direction.

Hence, on the hypothesis of circular vibrations, it is necessary that during its passage through an active medium, one of the rays should have got an advance of phase over the other equal to twice the angle of rotation of the plane of polarization. For flint glass and diamagnetic substances, the left circular rays gets in advance when the propagation is in the direction of the lines of force; the reverse is the case for magnetic media.

In any case the conception of two inverse circular rays which travel through a medium with different velocities is not a mere hypothesis; experiment shows, in fact, that the refractive index of a given circular ray differs according as it traverses the substance submitted to the action of magnetism in the direction of the lines of force, or in the contrary direction.

M. Cornu has shown, moreover, that in both cases the variations in the indices are equal and of opposite sign to that which the ray would have in the same medium when not under the action of magnetism.

598. This difference of phase may be variously explained; it may be assumed that the period is the same in both rays, but the velocity of propagation is different; or, that, while the velocity of propagation is the same, the period is no longer equal for the two rays, and is different for each of them from what it is in the external medium; or, lastly—and what is, perhaps, most probable, having regard to the ordinary laws of dispersion—that the period is modified as well as the velocity of propagation.

It is in general impossible to conceive a permanent vibratory state with a period different from that of the cause which produces it; but, in the present case, the difficulty does not seem to exist, if we assume that the medium which transmits light itself possesses a rotatory motion in a determinate direction; the period of the relative motion would be the same for both rays, and the same as in the external medium; the advance of phase would be due solely to the difference of the absolute motion, and precisely equal to the half of this difference.

In all cases when the two circular rays emerge from the medium, they assume the same period and the same velocity of propagation,

and they reconstitute a rectilinearly polarized ray. The difference of time that these two rays of velocities V′ and V″ take to traverse a thickness e of the medium is

$$t = e\left(\frac{1}{V''} - \frac{1}{V'}\right).$$

If T is the period of the ray in air, interference occurs on emergence between two rays whose difference of phase on entering was

$$(1) \qquad 2\pi\frac{t}{T} = 2\pi\frac{e}{T}\left(\frac{1}{V''} - \frac{1}{V'}\right).$$

As these rays have the periods T′ and T″ in the medium, the numbers m' and m'' of oscillations which they have made are

$$m''T' = \frac{e}{V'} \quad \text{and} \quad m''T'' = \frac{e}{V''},$$

which gives

$$m'' - m' = e\left(\frac{1}{V''T''} - \frac{1}{V''T'}\right).$$

The difference of phase on emergence is then

$$(2) \qquad 2\pi\left(m'' - m' + \frac{t}{T}\right) = 2\pi e\left[\frac{1}{V''}\left(\frac{1}{T''} + \frac{1}{T}\right) - \frac{1}{V'}\left(\frac{1}{T'} + \frac{1}{T}\right)\right],$$

and the rotation of the plane of polarization

$$(3) \qquad \theta = \pi e\left[\frac{1}{V''}\left(\frac{1}{T''} + \frac{1}{T}\right) - \frac{1}{V'}\left(\frac{1}{T'} + \frac{1}{T}\right)\right].$$

If V and λ denote the velocity, and the wave length of the ray in air, we may write

$$(4) \qquad \theta = \frac{\pi e}{\lambda}\left[\frac{V}{V''}\left(1 + \frac{T}{T''}\right) - \frac{V}{V'}\left(1 + \frac{T}{T'}\right)\right].$$

599. Whatever be the cause which modifies the circular vibrations, it is to be foreseen that the effects, being very small, become equal and of opposite signs on the two inverse rays, and that they are proportional to the magnetic force X.

Hence, if U is the velocity, and T the period of the ray in question, if the magnetic field were suppressed, and if n is the refractive index corresponding to the velocity U,

$$(5)\qquad \begin{aligned} \frac{V}{V'} &= \frac{V}{U}(1-\alpha X) = n(1-\alpha X), \\ \frac{V}{V''} &= \frac{V}{U}(1+\alpha X) = n(1+\alpha X); \end{aligned}$$

and

$$(6)\qquad \frac{T}{T'} = 1-\beta X, \quad \frac{T}{T''} = 1+\beta X.$$

The rotation of the plane of polarization becomes, in that case,

$$\theta = \frac{2\pi e n}{\lambda}(2\alpha+\beta)X.$$

The field being supposed uniform, the product eX represents the difference of potential at the entrance and emergence of the ray; we shall have then

$$(7)\qquad \omega = \frac{\theta}{eX} = \frac{2\pi n}{\lambda}(2\alpha+\beta).$$

For ordinary rotation, Fresnel assumes that the period does not change. If this be admitted, the coefficient β is null; if n' and n'' are the refractive indices of the two circular rays, we have simply

$$(8)\qquad \omega = \frac{2\pi}{\lambda X}\left(\frac{V}{V''}-\frac{V}{V'}\right) = \frac{2\pi}{\lambda}\frac{n''-n'}{X} = \alpha\frac{4\pi n}{\lambda}.$$

600. This would be the rotation if the velocity of propagation were independent of the wave length; but it must still be observed that the rotatory polarization brings about a dispersion between the two circular rays.

The velocities V′ and V″ correspond to two different lengths λ' and λ'', and the values of U and of n, which are introduced into the second term of the equation (4), refer to the propagation of the vibrations of these two lengths of wave in the medium in the natural state. Hence, restricting ourselves to terms of the first order, we may write

$$(5)' \qquad \begin{aligned} n' &= \left[n + \frac{dn}{d\lambda}(\lambda' - \lambda) \right] (1 - \alpha X) = n - n\alpha X + \frac{dn}{d\lambda}(\lambda' - \lambda), \\ n'' &= \left[n + \frac{dn}{d\lambda}(\lambda'' - \lambda) \right] (1 + \alpha X) = n + n\alpha X + \frac{dn}{d\lambda}(\lambda'' - \lambda); \end{aligned}$$

it follows from this that

$$(9) \qquad \omega = \alpha \frac{4\pi}{\lambda} \left[n + \frac{1}{2\alpha X} \frac{dn}{d\lambda}(\lambda'' - \lambda') \right].$$

We have, moreover,

$$\frac{\lambda''}{\lambda'} = \frac{V''}{V'} = \frac{1 - \alpha X + \frac{1}{n}\frac{dn}{d\lambda}(\lambda' - \lambda)}{1 + \alpha X + \frac{1}{n}\frac{dn}{d\lambda}(\lambda'' - \lambda)} = 1 - 2\alpha X + \frac{1}{n}\frac{dn}{d\lambda}(\lambda' - \lambda'');$$

which gives, to the same degree of approximation,

$$\frac{\lambda'' - \lambda'}{2\alpha X} = -\lambda \left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda} \right) = -\lambda.$$

The magnetic rotation then becomes

$$(10) \qquad \omega = \alpha \frac{4\pi}{\lambda} \left(n - \lambda \frac{dn}{d\lambda} \right).$$

We see that the result, independent of the dispersion, must be multiplied by the factor $(1 - \frac{\lambda}{n}\frac{dn}{d\lambda})$.

The coefficient α is itself a function of the wave length. To agree with the formula of (595) which best satisfies the experiments, the coefficient must be proportional to $\frac{n^2}{\lambda}$. We shall see from what considerations this formula may be theoretically deduced.

601. Observations of Sir W. Thomson.—According to Sir W. Thomson, the phenomena of magnetic rotatory polarization would appear to confirm Ampère's ideas on the ultimate nature of magnetism.

"The magnetic influence on light, discovered by Faraday, depends on the direction of motion of moving particles. For instance, in a medium possessing it, particles in a straight line perpendicular to the lines of magnetic force displaced to a helix round this line as axis, and then projected tangentially with such velocities as to describe circles, will have different velocities according as their motions are round in one direction (the same as the nominal direction of the galvanic current in the magnetising coil), or in the contrary direction. But the elastic reaction of the medium must be the same for the same displacements, whatever be the velocities and directions of the particles; that is to say, the forces which are balanced by the centrifugal force of the circular motions are equal, while the luminiferous motions are unequal. The absolute circular motions being therefore either equal, or such as to transmit equal centrifugal forces to the particles initially considered, it follows that the luminiferous motions are only components of the whole motion; and that a less luminiferous component in one direction, compounded with a motion existing in the medium when transmitting no light, gives an equal resultant to that of a greater luminiferous motion in the contrary direction compounded with the same non-luminous motion.

"I think it is not only impossible to conceive any other than this dynamical explanation of the fact, that circularly polarized light transmitted through magnetised glass parallel to the lines of magnetising force with the same quality, right-handed always, or left-handed always, is propagated at different rates according as its course is in the direction, or is contrary to the direction in which a north magnetic pole is drawn; but I believe it can be demonstrated that no other explanation of that fact is possible. Hence it appears that Faraday's optical discovery affords a demonstration of the reality of Ampère's explanation of the ultimate nature of magnetism; and gives a definition of magnetisation in the dynamical theory of heat.

"The introduction of the principle of moments of momenta ('the conservation of areas') into the mechanical treatment of Mr. Rankine's hypothesis of 'molecular vortices,' appears to indicate a line perpendicular to the plane of resultant rotatory momentum ('the invariable plane') of the thermal motions, as the

magnetic axis of a magnetised body, and suggests the resultant moment of momenta as the definite measure of 'magnetic moment.' The explanation of all phenomena of electromagnetic attraction or repulsion, and of electromagnetic induction, is to be looked for simply in the inertia and pressure of the matter of which the motions constitute heat. Whether this matter is or is not electricity—whether it is a continuous fluid interpermeating the spaces between molecular nuclei, or is itself molecularly grouped—or whether all matter is continuous, and molecular heterogeneousness consists in finite vortical or other relative motions of contiguous parts of a body, it is impossible to decide, and perhaps in vain to speculate, in the present state of science."—(*Reprint of Papers*, p. 419.)

Electrical Double Refraction.—It is known that whenever a singly refracting transparent body is subjected to a mechanical action, this body acquires for the moment double refracting properties, the axis of double refraction being directed along the line of pressure or of traction.

Kerr has shown that any singly refracting solid or liquid placed in an electrical field acquires a transient double refraction; the axis of double refraction coincides with the line of force, and according to the nature of the body, the velocity of the extraordinary ray is greater or less than that of the ordinary ray.

If δ is the intensity of double refraction—that is the difference of path between the ordinary and extraordinary ray for unit thickness of the dielectric—and if V is the difference of electrostatic potential between two points at a distance d, the electrical force F in the region in question is equal to $\frac{V}{d}$. Kerr deduces from his experiments the ratio

$$\delta = k\frac{V^2}{d^2} = kF^2,$$

k being a constant characteristic of the body, positive or negative according to circumstances. It follows from this that *the intensity of electrical double refraction is proportional to the square of the electrical force.*

We have seen (107) that the dielectric may be considered as subjected to a strain in the direction of the lines of force proportional to the square of the force. It appears then that the phenomenon observed by Mr. Kerr may be considered as an accidental double refraction, due to the electrostatic tension of the medium.

CHAPTER IX.

ELECTRICAL UNITS.

603. FUNDAMENTAL UNITS.—DERIVED UNITS.—In the phenomena of electricity and magnetism experimenters have for a long time evaluated the various quantities only as functions of arbitrary units, the choice of which was determined in each case by the convenience of experiments. This method, even when the units employed were suitably defined, has the inconvenience not only of making it very difficult to compare the results obtained by different observers, but particularly of masking the relations which may exist between various orders of phenomena. It is then of the greatest importance, for the progress of science, to arrive at a common understanding as to the choice of the units, and at the same time that the units adopted shall have that character of co-ordination amongst themselves which constitutes the superiority of the metrical system.

The units corresponding to the different kinds of magnitudes may, in fact, be chosen arbitrarily, and independently of each other; but there is an obvious advantage in making them depend on as small a number of simple units as possible. Thus, in geometry, the unit of surface and the unit of volume may be derived from the unit of length. In Kinematics we introduce with the velocity a new idea and a new unit—that of time. The study of dynamics leads to a third unit, independent of the two former—the unit of force or the unit of mass. All mechanical magnitudes may thus be evaluated as functions of the three units of length, of time, and of mass, or of length, of time, and of force. In a co-ordinated system, the irreducible units are called *fundamental units;* the others are called *derived units.*

All magnitudes which we deal with in electricity and magnetism have been defined by their mechanical properties; they may therefore be measured, like the mechanical quantities themselves,

as a function of the three fundamental units of length, of time, and of mass.

A system of measurements based on these principles is called an *absolute system*, the word absolute being employed in contradistinction to the term *relative*, which would characterize a system of measurements independent of each other.

604. DIMENSIONS OF A DERIVED UNIT.—Let n be the numerical expresssion of a quantity, that is the number of units which it contains, and [N] the magnitude of the unit of comparison; if this number be taken equal to [N′], the magnitude to be measured will be expressed by another number n', and we shall have the ratio

$$n[\mathrm{N}] = n'[\mathrm{N}'],$$

which gives

$$\frac{n'}{n} = \frac{[\mathrm{N}]}{[\mathrm{N}']}.$$

It follows from this that *the ratio of the numerical values of a given quantity is equal to the inverse ratio of the magnitudes which have served to measure it.*

When the unit is a derived unit, and it varies in consequence of a change in the magnitude of the fundamental unit, in order to learn this latter ratio we must know in what manner the derived unit depends on the fundamental units. The relation of a derived unit to the fundamental units determines the *dimensions* of this unit.

We shall represent the fundamental units of length, mass, and time, by the symbols [L], [M], and [T], and the magnitude of any unit by a letter enclosed within a square bracket $[x]$. The dimensions of the unit of surface will be represented by the symbol $[\mathrm{L}^2]$, and those of volume by the symbol $[\mathrm{L}^3]$; that is to say that the unit of surface varies as the square, and the unit of volume as the cube of the unit of length.

More generally if the dimensions of a derived unit are expressed by the symbol $[\mathrm{L}^p\mathrm{M}^q\mathrm{T}^r]$, and if we take the values L, M, T, and L′, M′, T′ successively, as fundamental units, the ratio of the derived units in the two systems will be

$$\frac{[\mathrm{N}']}{[\mathrm{N}]} = \left(\frac{\mathrm{L}'}{\mathrm{L}}\right)^p \left(\frac{\mathrm{M}'}{\mathrm{M}}\right)^q \left(\frac{\mathrm{T}'}{\mathrm{T}}\right)^r.$$

605. Derived Mechanical Units.—The principal derived units in mechanics are *velocity*, *acceleration*, *force*, *work* or *energy*.

Velocity [v].—Velocity v is the path traversed by a moving body in unit time, or the quotient of a length by a time. Hence, the dimensions of velocity will be expressed by the formula

$$[v] = [LT^{-1}].$$

Acceleration [γ].—Acceleration γ, is the ratio of the increase of velocity to the increase of time; it is therefore the quotient of a velocity by a time, and we have for the dimensions of the unit,

$$[\gamma] = [LT^{-2}].$$

Force [f].—Force f is the product of a mass by an acceleration, which gives

$$[f] = [LMT^{-2}].$$

Work, *Energy* [W].—Work or energy is the product of a force by a length; *vis viva*, which is a quantity of the same kind, is the product of a mass by the square of a velocity. In both cases we have

$$[W] = [L^2MT^{-2}].$$

The unit of force is that which acting on unit of mass for unit time imparts to it unit acceleration.

The unit of work is the work produced by unit of force, when its point of application is displaced in its own direction by unit length.

These two latter units are not those in ordinary use; the weight is commonly taken as unit of force: for instance a gramme or a kilogramme, and the kilogrammetre as unit of work. This amounts to choosing the unit of force, instead of the unit of mass, as the third fundamental unit. The choice of a weight like that of the kilogramme in the Archives at Paris, as the unit of force, has this inconvenience, that if this body, or any other equivalent, is moved to another part of the globe, its true weight will no longer represent the unit of force, in consequence

of the change of the intensity of gravity; the mass of a body, on the contrary, is an invariable quantity wherever it may be placed.

It is easy to see what is the ratio between these two units; the formula $p = mg$, in which m represents the mass of a body, and p its weight in a place where the acceleration of gravity is g, shows that if the mass of a body is unity, its weight will impart to it an acceleration equal to g, and is equivalent to g times the unit of force as defined above, the acceleration being expressed as a function of the length taken as fundamental unit.

Thus, if we take the metre and the mass of the kilogramme as units, the unit of force is $\frac{1}{9{\cdot}81}$ kilogramme, or about 100 grammes. With the kilogramme as unit of force, the unit of mass is that of a body weighing 9·81 kilogrammes.

606. Electrical and Magnetic Derived Units.—The most important electrical magnitudes are the *quantity of electricity*, the *strength of the electrical field*, the *potential* or the *electromotive force*, the *capacity*, the *strength of the current*, the *resistance*, etc. We have, in like manner, for the magnetic units the *quantity of magnetism*, the *intensity of the magnetic field*, the *magnetic strength of a shell*, etc. All these magnitudes are connected by the ratios which define them, and if one is given the others follow from it.

In order to have an absolute system, the quantity which serves as the starting-point must be measured directly in mechanical units. Thus, the quantity of electricity might be defined by Coulomb's law (7), or the quantity of magnetism by the corresponding law (293), or again the strength of the current, by Ampère's electrodynamical law (473). Out of this arise three systems of absolute measure, which are independent, and incompatible, in which the various units are differently connected with the fundamental units, and to which the names *electrostatic system*, *electromagnetic system*, and *electrodynamic system* have been given.

There is no theoretical reason for preferring one system to the others; two of them, however, possess a greater practical importance: these are the electrostatic and electromagnetic systems. The units of the electrodynamic system only differ, moreover, by a numerical factor from the corresponding electromagnetic units, and their applications are less simple. We shall restrict ourselves to the first two; and shall represent the quantities measured in electrostatic units by small letters, and expressions of the same magnitude in the electromagnetic system by capitals.

607. ELECTROSTATIC SYSTEM.—*Quantity of Electricity* $[q]$.—Coulomb's law gives, for the repulsion f exerted between two equal masses q at the distance d,

$$f=\frac{q^2}{d}, \qquad \text{or} \qquad q=d\sqrt{f}.$$

From which is deduced, for the dimensions of the unit of electricity,

$$[q]=[d][f^{\frac{1}{2}}]=[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}].$$

Surface density.—Electrical displacement $[\sigma]$.—The density is the quantity of electricity for unit surface (18); the displacement is the quantity which has traversed unit surface (126); we shall have then,

$$[\sigma]=[q][L^{-2}]=[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}].$$

Electrical force.—Strength of the field $[h]$. This is the force which acts on unit mass at the point in question, which gives for the dimensions of the unit,

$$[h]=[f][q^{-1}]=[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}];$$

they are the same as that of density, as could be foreseen from Coulomb's theorem (35).

The *flow of electrical force* is the product of electrical force into a surface; its dimensions are $[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$.

Specific inductive capacity $[k]$.—The specific inductive capacity is a number in the electrostatic system.

Electromotive force or *electrostatic potential* $[e]$.—The potential of an electrical mass at a distance d, is the quotient of the mass by this distance; we have then

$$[e]=L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}].$$

Electrostatic capacity $[c]$.—The capacity of a condenser is the quotient of its charge by the difference of potential of the armatures, and we have

$$c=\frac{q}{e}, \qquad \text{or} \qquad [c]=[L].$$

Electrostatic capacity is therefore a length, as we have already seen (73).

Strength of current $[i]$.—The strength of a current is the quantity of electricity which traverses the section of a wire in unit time, or the quotient of a quantity q, by the time which it takes to traverse this section. We have accordingly

$$i=\frac{q}{t}, \qquad \text{or} \qquad [i]=[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}].$$

Resistance $[r]$.—The resistance of a conductor is defined by Ohm's law (204). It is the quotient of the electromotive force between two points, by the strength of the current, which gives

$$r=\frac{e}{i}, \qquad \text{and} \qquad [r]=[L^{-1}T].$$

Electrostatic resistance is therefore the inverse of a velocity, as has already been proved (206).

Quantity of magnetism $[q']$.—In the electrostatic system, the quantity of magnetism is defined by the condition that the action of a magnetic pole of mass q', on a portion of current whose strength is i and whose length l is very small, at a distance d from the pole, and perpendicular to the right line which joins its centre to the pole, shall be defined by the elementary law (458)

$$f=\frac{ilq'}{d^2}, \qquad \text{or} \qquad q'=\frac{fd^2}{il};$$

from which is deduced

$$[q']=[L^{\frac{1}{2}}M^{\frac{1}{2}}].$$

Magnetic density $[\sigma']$.—The surface density being the quantity of magnetism for unit surface, we have

$$[\sigma']=[L^{-\frac{3}{2}}M^{\frac{1}{2}}].$$

Magnetic force.—Magnetic field [h'].—This is the force which acts on unit of magnetic mass; hence,

$$[h'] = [f][q'^{-1}] = [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}].$$

Magnetic potential [e'].—The magnetic potential is the work of the magnetic force—that is to say, the product of this force by a length; we shall have then

$$[e'] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}].$$

Strength of magnetic shell [ϕ].—The magnetic strength of a shell is the product of the surface density by the thickness of this shell, which gives

$$[\phi] = [L^{-\frac{1}{2}}M^{\frac{1}{2}}].$$

608. Electromagnetic System.—*Quantity of magnetism* [Q].—In the electromagnetic system, the starting-point of the measurements is the definition of the quantity of magnetism by Coulomb's law,

$$f = \frac{Q'^2}{d^2}; \qquad \text{where} \qquad [Q'] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}].$$

They are, of course, the same dimensions as those of the unit of electricity in the electrostatic system.

Surface magnetic density [Σ'].—*Magnetic force; strength of the field* [H'].—*Magnetic potential* [E'].—It follows from the preceding remark that the dimensions of these various units will be the same as those of the corresponding electrical quantities in the electrostatic system; that is to say,

$$[\Sigma'] = [H'] = [L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}],$$

$$[E'] = [L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}].$$

Strength of magnetic shell [Φ].—*Strength of the current* [I].—As the magnetic strength of a shell is the product of a surface density by the thickness, we have

$$[\Phi] = [L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}].$$

These dimensions are the same as those of potential, as could be foreseen (329).

The dimensions and the value of current strength, are the same as the magnetic power of a shell.

Quantity of electricity [Q].—The quantity of electricity being the product of a current strength by a time, we have

$$[Q] = [L^{\frac{1}{2}} M^{\frac{1}{2}}].$$

These dimensions are the same as those of the quantity of magnetism in the electrostatic system; hence, in the electromagnetic system, the *surface density*, the *force*, and the *electrical potential* will have the same dimensions as the corresponding magnetic quantities in the electrostatic system.

Specific inductive capacity [K].—The specific inductive capacity (126) is inversely as the coefficient of electrical elasticity of the medium—that is to say, proportional to the ratio of the displacement to the corresponding force; which gives

$$[K] = [L^{-2} T^2];$$

it is therefore equal to the inverse of the square of a velocity.

Resistance [R].—The resistance of a conductor may be defined by Joule's law (244), which gives

$$W = I^2 R t,$$

from which is deduced

$$[R] = [LT^{-1}].$$

The electromagnetic resistance is therefore a velocity; we have obtained this result directly (407). Suppose that the two rails, and the bar in the experiment assumed in this paragraph, are without

appreciable resistance, and that there is no other resistance in the circuit than that of the wire which connects the two points A and B. The resistance of this wire will be equal to the absolute unit, if the bar, being equal to unit length, and moving in unit field with unit velocity, perpendicular to the lines of force, gives rise to a current capable of producing in the wire unit of energy per second in the form of heat.

Electromotive force [E].—The electromotive force is deduced from Ohm's law

$$E = IR,$$

and its dimensions are

$$[E] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}].$$

Capacity [C].—Capacity being the ratio of the quantity of electricity which charges a condenser, to the difference of potential of the two armatures, we have again

$$C = \frac{Q}{E}, \qquad \text{or} \qquad [C] = [L^{-1} T^{2}].$$

609. Dimensions of the Principal Units.—We might in the same way determine the dimensions of the other quantities which we have not examined. We shall give in the following tables the dimensions of the most important quantities.

FUNDAMENTAL UNITS.

Length	[L],
Mass	[M],
Time	[T].

DERIVED MECHANICAL UNITS.

Velocity	$[LT^{-1}]$,
Acceleration	$[LT^{-2}]$,
Force	$[LMT^{-2}]$,
Energy	$[L^{2}MT^{-2}]$.

DERIVED ELECTRICAL UNITS.

	Electrostatic System.	Electromagnetic System.
Quantity of electricity	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$	$[L^{\frac{1}{2}}M^{\frac{1}{2}}]$
Electrical surface density Electrical displacement	$[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$	$[L^{-\frac{3}{2}}M^{\frac{1}{2}}]$
Electrical force Electrical field	$[L^{\frac{1}{2}}M^{-\frac{1}{2}}T^{-1}]$	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}]$
Flow of electrical force	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$	$[L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-2}]$
Specific inductive capacity	1	$[L^{-2}T^{2}]$
Electrostatic potential Electromotive force	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$
Electrostatic capacity	$[L]$	$[L^{-1}T^{2}]$
Strength of current	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Resistance	$[L^{-1}T]$	$[LT^{-1}]$

DERIVED MAGNETIC UNITS.

	Electrostatic System.	Electromagnetic System.
Quantity of magnetism	$[L^{\frac{1}{2}}M^{\frac{1}{2}}]$	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$
Surface density	$[L^{-\frac{3}{2}}M^{\frac{1}{2}}]$	$[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Magnetic force Magnetic field	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}]$	$[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Flow of magnetic force	$[L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-2}]$	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$
Magnetic potential	$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Magnetic power	$[L^{-\frac{1}{2}}M^{\frac{1}{2}}]$	$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Magnetic moment	$[L^{\frac{3}{2}}M^{\frac{1}{2}}]$	$[L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-1}]$
Strength of magnetisation	$[L^{-\frac{3}{2}}M^{\frac{1}{2}}]$	$[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$
Coefficient of magnetisation Magnetic permeability	$[L^{-2}T^{2}]$	1
Verdet's constant	$[L^{-\frac{3}{2}}M^{-\frac{1}{2}}T^{2}]$	$[L^{-\frac{1}{2}}M^{-\frac{1}{2}}T]$
Coefficients of mutual induction and of self-induction	$[L^{-1}T^{2}]$	$[L]$

610. Relations between the Two Systems of Units.—In order to establish a relation between the corresponding units, we may compare their dimensions, or equalise the numerical expression of the same quantity as a function of each of them. Consider, for instance, the different expressions for the same quantity of energy W; we shall have the equalities .

$$\begin{aligned} W &= i^2rt = I^2Rt, \\ W &= eit = EIt, \\ W &= eq = EQ, \\ W &= e^2c = E^2C. \end{aligned}$$

From this we deduce the constant value a,

$$a = \sqrt{\frac{R}{r}} = \frac{i}{I} = \frac{E}{e} = \frac{q}{Q} = \sqrt{\frac{c}{C}}.$$

These being the ratios between the numerical values, we shall have for the ratios of the corresponding units (604)

$$\frac{[I]}{[i]} = \frac{[e]}{[E]} = \frac{[Q]}{[q]} = a,$$

$$\frac{[r]}{[R]} = \frac{[C]}{[c]} = a^2.$$

The constant a denotes then the number of electrostatic units $[q]$ of electricity which there are in an electromagnetic unit [Q].

Since the electromagnetic resistance R is a velocity, and the electrostatic resistance r is the inverse of a velocity, the ratio $\frac{R}{r}$ or $\frac{[r]}{[R]}$ is the square of a velocity. As this ratio is equal to a^2, it follows that the constant a is itself a velocity.

A great many experiments have been made in order to determine the value of this constant. There are clearly as many methods as there are quantities which can be measured in electrostatic as well as in electromagnetic units. All the results obtained range about the number which expresses the velocity of light in air. It is probable that this is not an accidental coincidence, and that the equality of the two numbers arises from a correlation in the

nature of the phenomena. The velocity of light is very approximately 300,000 kilometres per second. This is the number which we shall take for the ratio a, expressing it as a function of the length which has been chosen as fundamental unit.

611. Choice of Fundamental Units.—The choice of units adopted as units of time, of length, and of mass, is obviously arbitrary. As unit of time the second, the sixtieth of the minute of the mean time adopted by all civilised peoples, naturally suggests itself; as unit of length we may take either the metre or a decimal of a metre. As unit of mass it is advantageous to take the mass of unit volume of water at its greatest density; we thus retain the advantage that the specific gravity of water is equal to unity, and that the weight of a body is equal to the product of its volume by its density.

The absolute system which would be least foreign to the habitudes established in the use of the metrical system, would consist in taking the decimetre and the mass of a kilogramme as fundamental units.

Gauss and Weber, who introduced into science the first absolute system, had chosen the millimetre and the mass of a milligramme. The British Association adopted the centimetre and the gramme on the proposition of Sir W. Thomson. These latter units were definitely adopted for electrical and magnetic measurements by the International Congress of Electricians, which met at Paris in 1881.

612. Absolute C.G.S. System.—It has been agreed that the units derived from the centimetre, from the mass of a gramme, and from the second of mean time, shall form the absolute system, properly so-called, and which will be denoted by the symbol C.G.S.

These units have not received any special names. Thus the unit of force C.G.S. is the force which, acting on the mass of a gramme, imparts to it in a second the acceleration of a centimetre. It follows from this that a gramme is g units of force C.G.S., and a kilogramme $g.10^3$ C.G.S. units, g being expressed in centimetres. In like manner, a kilogrammetre is $g.10^5$, that is to say, 981.10^5, or about 10^8 C.G.S. units of work. On this system the value of a is 3.10^{10}.

613. Practical System.—The values of the absolute units of the C.G.S. unfortunately do not stand in any convenient relation to the magnitudes we have to measure in practice. Thus, the absolute unit of the C.G.S. electromagnetic resistance is scarcely the resistance of a twenty-millionth of a millimetre of copper wire a millimetre in diameter, and the unit of electromotive force would be the one-hundred-millionth part of that of a Daniell.

Accordingly the Committee of the British Association, founded in 1861 for the purpose of establishing a rational system of electrical measurements, was led to choose units more suited for practical needs, and to give special names to these units so as to facilitate their adoption. This system has been ratified in the following form by the Congress at Paris:

The practical unit of resistance is equal to 10^9 C.G.S. absolute units, and acquires the name of *Ohm.**

The *Volt* is the practical unit of electromotive force; it is equal to 10^8 C.G.S. units.†

The *Ampère* is the current produced by the electromotive force of a volt in a circuit having the resistance of an ohm; it is equal to 10^{-1} C.G.S.

The *Coulomb* is the quantity of electricity which, in a second, traverses the section of a conductor which is conveying the current of an ampère; the coulomb is equal to 10^{-1} C.G.S.

The *Farad* is the capacity of a condenser whose armatures acquire a difference of potential of one volt when the charge is a coulomb; the farad is 10^{-9} C.G.S. units.

In certain applications it is useful to express different magnitudes by means of units which are a million times as small or as great as the corresponding practical unit. These new units are called by the same name with the prefix *mega* or *micro*, according as they are multiplied or divided by a million,

Thus the *megohm* is equal to 10^6 ohms; the submultiple called the *microhm* is 10^{-6} ohms. In like manner, for capacity, the millionth of the farad, or the *microfarad*, is equal to 10^{-6} farads or 10^{-15} C.G.S. units.

* The Committee of the British Association made numerous experiments in order to determine the value of the Ohm and to construct material standards presenting the same resistance. The first investigations appeared to show that the Ohm is represented with a close degree of approximation by the resistance of a column of mercury at 0°, a square millimetre in cross section, and 104 centimetres in length; but it appears that certain errors were made in the calculation, and that this length should be increased by about one per cent.—that is, raised to 105 centimetres. Recent investigations lead to the same result, but the question does not seem to have been definitely solved. As it is not certain, on the other hand, that solid metals do retain their electrical properties without change, the Congress decided that the Ohm should be represented by a column of mercury at zero, having a cross section of one square millimetre, and that an international commission should settle by fresh experiments the exact length of this unit.

† The electromotive force of a Daniell with sulphuric acid is about 1.08 volts.

The microfarad is really the practical unit of capacity, for the value of the farad is far too great. For instance, the electrostatic capacity of the Earth is equal to its radius R; its electromagnetic capacity C is equal to the quotient of the radius by a^2, which gives

$$C = \frac{R}{a^2} = \frac{2 . 10^9}{\pi} \frac{1}{3^2 . 10^{20}} = 708 . 10^{-15} \text{ C.G.S. units},$$

that is to say, 708 microfarads.

It is important to observe that the practical units themselves constitute an absolute system, in which the fundamental units are

$[L] = 10^7$ metres, or the quarter of the terrestrial meridian,
$[M] = 10^{-11}$ of the mass of a gramme,
$[T]$ = a second.

Another remark, which is of some practical utility, is, that if we divide the expression for electrical work by g expressed in metres—that is to say, practically by 10—we get its value in kilogrammetres.

The practical unit of work is obtained, for instance, if we multiply a volt by an ampère, which gives 10^7 C.G.S. units. But we have seen (612) that a kilogrammetre is equivalent to 10^8 C.G.S. units—that is to say, to ten times as much. The same quantity of work in kilogrammetres will then be expressed by one-tenth of that value.

614. COMPARATIVE VALUES OF THE PRINCIPAL UNITS.—We shall collate in the following table the values of the practical units in C.G.S. absolute units, and also as functions of the units of Gauss and Weber, which have been employed in a certain number of papers on electricity.

FUNDAMENTAL UNITS.	PRACTICAL UNITS.	C.G.S. UNITS.	UNITS OF GAUSS AND WEBER.
Length,	10^7 metres,	Centimetre,	Millimetre,
Mass,	10^{-11} gramme,	Gramme,	Milligramme,
Time.	Second.	Second.	Second.
Resistance	Ohm	10^9	10^{10}
Electromotive force	Volt	10^8	10^{11}
Current	Ampère	10^{-1}	10
Quantity	Coulomb	10^{-1}	10
Capacity	Farad	10^{-9}	10^{-10}

615. PHYSICAL CONCEPTION OF THE VELOCITY a.—We may give a physical representation of the velocity a which expresses the ratio of the electrical units in the two systems.

Suppose, for instance, that a sphere of radius R, or a condenser of the same capacity, were charged in such a manner that its electrical potential is equal to unity, and is discharged n times in the time t through a conductor; the mean strength of the current in electrostatic units will be $\frac{n\mathrm{R}}{t}$. If we determine n in such a manner that the strength of this current is equal to the electromagnetic unit, the expression $\frac{n\mathrm{R}}{t}$ will represent the number of electrostatic units of electricity which are contained in an electromagnetic unit; that is to say, the value of a, and this expression is a velocity.

616. Maxwell points out another mode of representing this, based on the hypothesis that the external action of an electrical mass in motion is equivalent to that of a current.

Consider a plane covered with a uniform charge of electricity of density σ, and moving in its own plane with a velocity u.

Each band of unit breadth, and parallel to the direction of the motion, is the equivalent of a current whose intensity is σu in electrostatic measure, and $\frac{\sigma u}{a}$ in electromagnetic measure. Suppose, now, that a second plane parallel to the first, at a distance δ, moves in the same manner, and in the same direction, with a velocity u', and let σ' be its density. Two kinds of actions are produced between these planes; an electrostatic repulsion in virtue of charges of the same kind, and an electrodynamic attraction due to parallel currents in the same direction.

Let us now take in the second plane a band of length l, and of infinitely small breadth b, and in the former plane an unlimited band of breadth dx, at a distance x, from the projection of the band bl. The electromagnetic action exerted by this unlimited band on the first situated at the distance $\sqrt{\delta^2+x^2}$, is expressed by (480)

$$2\frac{\sigma u}{a}dx\frac{\sigma' u'}{a}b\frac{l}{\sqrt{\delta^2+x^2}}=2\frac{\sigma\sigma' uu'}{a^2}bl\frac{dx}{\sqrt{\delta^2+x^2}},$$

and its component df along the perpendicular to the plane is

$$df=2\frac{\sigma\sigma' uu'}{a^2}bl\frac{\delta dx}{\delta^2+x^2}.$$

In order to have the total action of the first plane on the movable portion in question bl of the second, we must integrate this expression from $x = -\infty$ to $x = +\infty$, which gives

$$f = 2\frac{\sigma\sigma' uu'}{a^2} bl \int_{-\infty}^{+\infty} \frac{\delta dx}{\delta^2 + x^2} = 2\frac{\sigma\sigma' uu'}{a^2} bl \left[\tan\frac{x}{\delta}\right]_{-\infty}^{+\infty},$$

or

$$f = 2\pi\frac{\sigma\sigma' uu'}{a^2} bl.$$

On the other hand, the electrostatic charge of this surface is $bl\sigma'$. As the action of the former unlimited plane on the unit of mass is equal to $2\pi\sigma$, the repulsion f' exerted on this surface is perpendicular, and its value is

$$f' = 2\pi\sigma\sigma' bl.$$

If these two actions are equal, the same would be the case for all other portions of the second plane, and there would be equilibrium between them. For this uu' must be equal to a^2, or, if the velocities u and u' are equal, $u = a$.

The constant a is therefore such that if two unlimited parallel planes, uniformly electrified, moved in the same direction with this velocity, their electrodynamic action would counterbalance their electrostatic repulsion. As the velocity a is that of light, the experiment cannot be realised in this form.

617. In order to evaluate the order of magnitude of the effects which may be obtained, we may observe that an unlimited band of breadth b, and density σ, movable in its own direction with a velocity u, is equivalent to a current whose electromagnetic intensity is $\frac{u\sigma}{a}b$. If we suppose it placed at a distance e from a similar band, and if the condenser thus formed be charged to an electrostatic potential V, we shall have (74)

$$\sigma = \frac{V}{4\pi e}.$$

Now, by means of electrical machines, we can get potentials equal to 100,000 Daniell's cells—that is to say about 10^5 volts, or

$10^5 . 10^8$ C.G.S. units. With such machines, we shall have, in electrostatic units,

$$V = \frac{10^5 . 10^8}{a} = \frac{10^{13}}{3 . 10^{10}} = 333 ,$$

which gives virtually $\sigma = \frac{30}{e}$.

If we assume $b = 10$ and $e = 1$, the electromagnetic intensity of the current will be

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$$I = \frac{u . 300}{3 . 10^{10}} = \frac{u}{10^8} \text{ C.G.S.}$$

As a volt in a circuit of n ohms, gives a current of strength

$$I' = \frac{1}{10n} \text{ C.G.S.},$$

it will be seen that in order to have the same current the band must move with a velocity,

$$u = \frac{10^7}{n} = \frac{100000}{n} \text{ metres.}$$

It must be observed that the current I, produced by the motion of an electrical body, is far more difficult to observe than that of an ordinary electrical cell, since it must act directly on the needle, and without any effect of multiplication.

Rowland has proved experimentally that the rotation of an electrified disc produces a sensible effect on a magnetised needle, and that the action is of the same order as that which would be given by the preceding considerations.

CHAPTER X.

GENERAL THEORIES.

618. AMPÈRE'S HYPOTHESES.—In order to establish the elementary formula of electrodynamic actions, Ampère relied solely on the hypothesis of central forces, and on certain experimental facts, without any particular view as to the nature of electrical currents themselves. Nevertheless, from the year 1822, he "endeavoured to account for the force which is exerted between two elements of conducting wires, by the action of the fluid, which is distributed in space, and whose vibrations produce the phenomena of light." Ampère pointed out another mode of conceiving the phenomena: "If we suppose that molecules of electricity put in motion in conducting wires by the action of the battery, are continually changing their places, uniting every moment to form neutral fluids, separating, and then quickly uniting with other molecules of the fluid of the opposite kind, it is not contradictory to assume that from the actions which are inversely as the squares of the distances separating the molecules, a force might be produced between two elements of conducting wires, which depends not only on their distance, but also on the directions of the two elements along which the electrical molecules are moving, uniting with molecules of the opposite kind, and then separating the moment after to unite with others."—*Mémoires de l'Institut* for 1823, pp. 294 and 299. Ampère did not follow the development of these ideas any further; he did not think the time had come to do it with utility.

The hypotheses of Ampère have been taken up from time to time by various physicists, particularly by Weber and by Maxwell: we shall give a summary of the theories proposed.

619. FORMULÆ OF GAUSS AND OF WEBER.—If we bring into play the reciprocal actions of the electrical masses which circulate in conductors, the action of the two electrical masses m and m' must be

a function not only of the distance r, but also of their relative motion, and the problem thus stated appears indeterminate. One hypothesis is to assume that this action, while directed along the right line joining them, and proportional to the product of the masses, and inversely as the square of the distance, comprises a term proportional to a power of the relative velocity u of the two masses, and another term proportional to a power of their relative velocity $\frac{dr}{dt}$ parallel to their distance. These powers must be even, if the action is not to be modified when both the directions of the two motions and of the currents themselves are changed, and the problem is satisfied by taking them equal to 2, which gives the elementary law

$$(1) \qquad \frac{mm'}{r^2}\left[1+\alpha u^2+\beta\left(\frac{dr}{dt}\right)^2\right].$$

Weber examined first the simple cases of two elements in the prolongation of each other, or perpendicular to the same right line. In the former case, it must be assumed that the action of the two masses contains a term which depends on their relative velocity, and he supposed that this term was proportional to the square of the velocity. The second case led Weber to bring in the acceleration along the same right line, and the simplest hypothesis was to assume that this fresh term was proportional to the acceleration. From this follows another elementary law

$$(2) \qquad \frac{mm'}{r^2}\left[1+\alpha'\left(\frac{dr}{dt}\right)^2+\beta'\frac{d^2r}{dt^2}\right].$$

It remains to determine the coefficients α, β, α' and β', which enter into these two expressions (1) and (2).

620. Consider, in the first place, two bodies moving respectively on two fixed curves s and s', with the constant velocities v and v'. When the two moving bodies traverse the elements ds and ds', which are at the distance r, and make with each other an angle ϵ, we have

$$u^2=v^2-2vv'\cos\epsilon+v'^2,$$

$$v=\frac{ds}{dt} \qquad \text{and} \qquad v'=\frac{ds'}{dt}$$

From this is deduced

$$\frac{dr}{dt}=\frac{\partial r}{\partial s}\frac{ds}{dt}+\frac{\partial r}{\partial s'}\frac{ds'}{dt}=v\frac{\partial r}{\partial s}+v'\frac{\partial r}{\partial s'},$$

$$\frac{d^2r}{dt^2}=v^2\frac{\partial^2 r}{\partial s^2}+2vv'\frac{\partial^2 r}{\partial s\partial s'}+v'^2\frac{\partial^2 r}{\partial s'^2}.$$

If we suppose that the two curves s and s' convey currents of strengths I and I', then from Ampère's hypothesis (351), and considering the force to be repulsive, the action of two elements ds and ds' is

$$(3)\qquad -\frac{2\mathrm{II}'dsds'}{r^2}\left[\cos\epsilon+\frac{3}{2}\frac{\partial r\partial r}{\partial s\partial s'}\right]=\frac{2\mathrm{II}'dsds'}{r^2}\left[r\frac{\partial^2 r}{\partial s\partial s'}-\frac{1}{2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right].$$

621. Suppose now that the element ds contains electrical masses m and m_1, moving respectively with the velocities v and v_1, and that in like manner the element ds' contains masses m' and m'_1, with the velocities v' and v'_1. If we evaluate the actions of the masses m and m_1 on the masses m' and m'_1, from the formulæ (1) and (2), the resultant should reproduce Ampère's laws in one or other of the two forms, and therefore only contains terms in which the product vv' of the velocities is a factor.

The terms containing the squares of the velocities are, to within a factor,

$$(mv^2+m_1v_1^2)(m'+m'_1), \qquad \text{and} \qquad (m'v'^2+m'_1v'^2_1)(m+m_1).$$

In order that these terms may be null, we must have

$$(4)\qquad mv^2+m_1v_1^2=0, \qquad \text{or} \qquad m'+m'_1=0.$$

These two conditions are realised simultaneously, if we assume with Weber that an electrical current of strength I is formed by two currents of contrary electricities, moving with the same velocity v in opposite directions, and each having one-half the intensity. It is even necessary to assume that the algebraical sum of the electrical masses which exist in each element of current in the permanent state is null, if the condition is to be satisfied (203) that the density in the conductor is null; but, without stating anything definite on the ratio of electrical masses of opposite signs, it would be sufficient that the sum $mv^2+m_1v_1^2$ were null in each element—that is to say that there were electrical masses of contrary signs with the same vis viva.

The quantity of electricity which traverses the section of the first conductor in unit time is equal to $mv + m_1v_1$; if a denotes the number of electrostatic units in the electromagnetic unit, we have

$$(5) \qquad \begin{aligned} mv + m_1v_1 &= a\mathrm{I}ds, \\ m'v' + m'_1v'_1 &= a\mathrm{I}'ds'. \end{aligned}$$

Assuming the existence of equal and opposite currents, these equations become

$$(6) \qquad \begin{aligned} 2mv &= a\mathrm{I}ds, \\ 2m'v' &= a\mathrm{I}'ds'. \end{aligned}$$

When we evaluate the action of the masses m and m_1 on the masses m' and m'_1, the terms which depend on the product of the velocities are, on the first hypothesis,

$$\frac{2\beta}{r^2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}(mv + m_1v_1)(m'v' + m'_1v'_1) - \frac{2a}{r^2}(mv + m_1v_1)(m'v' + m'_1v'_1)\cos\epsilon,$$

that is to say,

$$-\frac{2\mathrm{II}'dsds'}{r^2}\left[a^2\alpha\cos\epsilon - a^2\beta\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right];$$

and with the hypothesis of Weber,

$$2\frac{(mv + m_1v_1)(m'v' + m'_1v'_1)}{r^2}\left[\alpha'\frac{\partial^2 r}{\partial s\partial s'} + \beta'\frac{\partial r}{\partial s}\frac{dr}{ds'}\right],$$

$$\frac{2\mathrm{II}'dsds'}{r^2}\left[a^2\beta'\frac{\partial^2 r}{\partial s\partial s'} + a^2\alpha'\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right].$$

The term, which is independent of the relative motion, is equal in both cases to $\frac{(m + m_1)\ (m' + m'_1)}{r^2}$. This term should be null if there are two equal currents in opposite directions in each wire; if not it will represent an electrostatic action between the conductors, a phenomenon which up to the present time experiment has not ascertained to exist.

In order to satisfy Ampère's law, it is necessary that, in the first case,

$$a^2\alpha = 1, \qquad a^2\beta = -\frac{3}{2},$$

and, in the second case,

$$a^2\beta' = r, \qquad a^2\alpha' = -\frac{1}{2}.$$

The expressions (1) and (2), which give the elementary action of the two electrical masses, become then

$$(1)' \qquad \frac{mm'}{r^2}\left[1+\frac{1}{a^2}\left(u^2-\frac{3}{2}\left(\frac{dr}{dt}\right)^2\right)\right],$$

$$(2)' \qquad \frac{mm'}{r^2}\left[1+\frac{1}{a^2}\left(r\frac{d^2r}{dt^2}-\frac{1}{2}\left(\frac{dr}{dt}\right)^2\right)\right]=mm'\left[\frac{1}{r^2}+\frac{2}{a^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2}\right].$$

622. The former expression (1)', which occurs in Gauss's manuscripts, is incompatible with the principle of the conservation of energy, for it would lead to the conclusion that a limited physical system can produce an indefinitely increasing quantity of energy.

Weber's formula, on the contrary, is compatible with this principle; for the expression of the force (2)' may be considered as the differential coefficient in respect of r, taken with the contrary sign, of the function

$$(7) \qquad \psi = \frac{mm'}{r}\left[1-\frac{1}{2a^2}\left(\frac{dr}{dt}\right)^2\right].$$

The work done by the repulsion of a fixed mass on a movable mass is equal to the difference $\psi_0 - \psi_1$ of the values of the function ψ relative to these two masses, for the initial and the final position. The function ψ may be considered as representing the potential energy of the system of the two masses; it only depends on their distance, and on their relative velocity along the right line r; it resumes the same value when one of the masses describes a closed path in reference to the other, and possesses the same velocity at the same points.

Since induction is a consequence of the law of Ampère, and of the principle of the conservation of energy, Weber's formula, which equally well satisfies both conditions, must give the induction.

623. This formula also enables us to obtain directly the relative potential energy of two closed circuits. For, if we replace $\frac{dr}{dt}$ by its value as a function of the velocities of the electrical masses, we get, from Weber's hypothesis, that the potential of one of the elements on the other is expressed by

$$\psi(ds,ds') = -\Pi' ds ds' \frac{1}{r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}.$$

The potential energy of the two circuits is then

$$(8) \qquad W = -\Pi' \int\int \frac{1}{r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} ds ds' = \Pi' \int\int \frac{\cos\epsilon}{r} ds ds'.$$

This is the formula of Neumann, as found above (353).

When the currents are constant, and traverse circuits of constant form, the resultant of the actions exerted by one of the currents on any mass m' whatever of the other is perpendicular to its trajectory.

624. PHENOMENA OF INDUCTION.—Let us now suppose that the currents move, and that the strengths change. The distance r, instead of being, as above, simply a function of two independent variables s and s', is further a function of the time, and we have $r = \phi(s,s',t)$. For a given value of t, the function ϕ represents the distance of the two elements of the circuit; if t is variable, and we consider s and s' as functions of t, the value of ϕ represents the distance of two electrical masses in motion on movable conductors. If the conductor has a constant section, the velocities do not at all depend on s and s' as independent variables, for at each instant the strength is the same in all parts of the circuit. Hence, for the relative velocity of the two masses, we shall have

$$\frac{dr}{dt} = v\frac{\partial r}{\partial s} + v'\frac{\partial r}{\partial s'} + \frac{\partial r}{\partial t};$$

and, considering the differential coefficients $\frac{\partial r}{\partial s}$ and $\frac{\partial r}{\partial s'}$ as functions of s and s' alone, the relative acceleration will be

$$\frac{d^2r}{dt^2} = v^2\frac{\partial^2 r}{\partial s^2} + 2vv'\frac{\partial^2 r}{\partial s \partial s'} + v'^2\frac{\partial^2 r}{\partial s'^2} + \frac{\partial v}{\partial t}\frac{\partial r}{\partial s} + \frac{\partial v'}{\partial t}\frac{\partial r}{\partial s'} + v\frac{\partial v}{\partial t}\frac{\partial r}{\partial s} + v'\frac{\partial v'}{\partial s'}\frac{\partial r}{\partial s'} + \frac{\partial^2 r}{\partial t^2}.$$

The velocity $\frac{dr}{dt}$ and the acceleration $\frac{d^2r}{dt^2}$ refer to the electrical masses, while the terms $\frac{\partial r}{\partial t}$ and $\frac{\partial^2 r}{\partial t^2}$ of the second member refer to the distances of the two elements of the two circuits.

The mechanical action of ds on ds' will be obtained as above by taking the *sum* of the actions which the masses of the elements ds exert on those of the elements ds'. With Weber's formula and hypothesis, it is easily seen that in this sum there only remain, as before, terms in vv', and with coefficients already found. It follows from this, that in the variable state, the mechanical action is at each instant conformable with that which Ampère's formula would give.

625. The electromotive force which acts on the element ds' is the force which tends to separate the equal masses of opposite signs contained in this element, and to carry them in opposite directions. We shall obtain the value by taking the *difference* of the actions exerted in the direction of the element ds', on each of the masses which it contains, by the two masses of the elements ds. But when we add the actions of the two masses $+m$ and $-m$ of the element ds on one of the masses m' of the element ds, the terms which remain are terms in v, vv' and $\frac{dv}{dt}$, which change sign at the same time as m. Among these the only ones which remain in the final difference are those which change sign with the velocity v, whatever may be the sign of v'. These terms reduce to two: one arising from $\left(\frac{dr}{dt}\right)^2$, and which is $2v\frac{\partial r}{\partial s}\frac{\partial r}{\partial t}$, the other arising from $\frac{d^2r}{dt^2}$, which is $\frac{\partial v}{\partial t}\frac{\partial r}{\partial s}$.

The difference thus calculated is equal to

$$\frac{4mm'}{a^2r^2}\left[r\frac{\partial v}{\partial t}\frac{\partial r}{\partial s}-v\frac{\partial r}{\partial s}\frac{\partial r}{\partial t}\right]=\frac{1}{r^2}\left[r\frac{\partial \mathrm{I}}{\partial t}\frac{\partial r}{\partial s}-\mathrm{I}\frac{\partial r}{\partial s}\frac{\partial r}{\partial t}\right]dsds',$$

taking into account equations (6), and supposing the intensity equal to unity in the element ds'.

We must take the component of this action along ds', and therefore multiply the preceding expression by $\frac{\partial r}{\partial s'}$; observing that we have

$$r\frac{\partial \mathrm{I}}{\partial t}-\mathrm{I}\frac{\partial r}{\partial t}=r^2\frac{\partial}{\partial t}\left(\frac{\mathrm{I}}{r}\right),$$

the elementary electromotive force becomes

$$d^2\mathrm{E} = \frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\frac{\partial}{\partial t}\left(\frac{\mathrm{I}}{r}\right) ds ds'.$$

The total electromotive force produced in the circuit s' by the circuit s, is obtained if we integrate this expression in reference to s and to s'. As the intensity I is merely a function of the time, and as the limits of the integral are themselves independent of the time,

$$\mathrm{E} = \frac{d}{dt}\mathrm{I}\int\int \frac{1}{r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} ds ds'. \tag{9}$$

We shall have then (353)

$$\mathrm{E} = -\frac{d}{dt}\mathrm{I}\int\int \frac{\cos\epsilon}{r} ds ds' = -\frac{d}{dt}(\mathrm{IM}) = \frac{d\mathrm{Q}}{dt}, \tag{10}$$

which gives the general expression (518) of the electromotive force produced in a circuit by an external current. We shall find in like manner the other cases of induction.

626. VARIOUS ATTEMPTS AT A THEORY.—Numerous attempts have been made, after the example of Weber, to bring under one and the same theory the phenomena of statical electricity, of permanent currents, and the effects of induction, and to establish a connection between electricity, magnetism, and light.

Gauss expressed the opinion that electrical actions cannot take place instantaneously, and that we should have a key to electrodynamical phenomena, if we could discover the law of the propagation of electrical forces.

Guided by these considerations, several mathematicians have treated the problem. For instance, the phenomena of induction may be explained by assuming that the electrical potential is propagated in a medium with a certain velocity which would be the same as that of light, according to B. Riemann, or of a totally different order, according to the theory of C. Neumann.

M. Betti compares the action of currents to that of a system of elementary magnets tangential at every point to the contour of the circuit, and periodically polarized in opposite directions, and he considers the magnetic force as transmitted in the medium with a certain velocity.

M. Lorenz has shown that by adding, to the equations given by Kirchhoff for electrical currents, suitably chosen terms, which do not at all affect any experimental conclusion, we get a new series of equations which indicate an action from layer to layer in the medium, and a phenomenon of undulation travelling with the velocity of light. He thus arrives at results similar to those which Maxwell had deduced from an entirely different theory.

M. Edlund has attempted to show that electrical phenomena, both statical and dynamical, may be explained by the aid of a single fluid, which in all probability is nothing but the ether.

M. Edlund assumes that all bodies in the neutral state contain a normal quantity of ether, and that a positive or negative electrification corresponds to a share of ether, greater or less than that of the normal charge. It is easy to deduce from this that the action of two bodies is proportional to the excess of their respective charges over the normal charges.

An electrical current is then only a transport of ether in a given direction; if we assume that the action of the two masses only depends on their velocity and on their relative acceleration along the right line joining them, then by reasoning analogous to that of Weber, and determining certain coefficients by the identification of the formulæ with the results of experiment, we arrive at an explanation of Ampère's laws and the phenomena of induction.

All the preceding theories imply the existence of an intermediate medium; for if any mechanical effect, force, or potential, is transmitted with a finite velocity from one particle to another, it follows that a medium of a suitable structure must have been the seat of this action while this effect had quitted the first particle and had not yet reached the second. Maxwell has taken the properties of this medium into account, and has thus established remarkable numerical relations between the phenomena of electricity and of light, which are supported by experiment.

627. Electromagnetic Theory of Light.—We have seen on several occasions how favourable the various phenomena of electricity and magnetism are to Faraday's conception, which consists in giving up the idea of actions at a distance, and considering forces as transmitted by the elastic reaction of an intermediate medium. This is a hypothesis which at the present day forms the basis of the physical theory of light, but it would be contrary to the spirit of science to assume that there are as many different media as there are phenomena to explain, as was formerly done by the distinct hypotheses of calorific fluid, of electrical fluids, and of magnetic fluids.

The great problem which the philosophy of science raises is to know the constitution of the single medium by which all physical phenomena may be explained. If calculation shows that electromagnetic phenomena are propagated not only in air, but in all bodies, with the velocity of the propagation of light, the question would have made a great step; for it would be shown that this medium exists, and that in all probability electrical and luminous phenomena are only different manifestations of the properties with which it is endowed. Such is the conclusion from Maxwell's theory. Faraday's discovery of the action of a magnetic field on the polarization of the light which traverses it, would be a natural consequence of the connection which the common medium establishes between the two orders of phenomena.

628. GENERAL EQUATIONS.—In order to determine the conditions of the propagation of an electromagnetic disturbance in a medium, we shall suppose this medium at rest—that is to say, not subject to any other motion than that resulting from the disturbance itself.

Equations (11), (13), and (14) of (572), give

$$u' = u + \frac{\partial f}{\partial t},$$
$$4\pi f = \mathrm{K} P,$$
$$u = Pc.$$

From which is deduced

$$u' = cP + \frac{\mathrm{K}}{4\pi}\frac{\partial P}{\partial t},$$

an equation which may be written in the symbolical form

$$u' = \left(c + \frac{\mathrm{K}}{4\pi}\frac{\partial}{\partial t}\right) P. \tag{11}$$

The medium being fixed, the differentials of the co-ordinates in respect of time are null. Equations (10) of (571) give then

$$P = -\frac{\partial \mathrm{F}}{\partial t} - \frac{\partial \psi}{\partial x};$$

from this follows

$$4\pi u' = -\left(4\pi c + \mathrm{K}\frac{\partial}{\partial t}\right)\left(\frac{\partial \mathrm{F}}{\partial t} + \frac{\partial \psi}{\partial x}\right). \tag{12}$$

On the other hand, we have by equations (12) of (572),

$$4\pi u' = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \tag{13}$$

and the equations (2) of (567) give

$$\mu\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) - \Delta F = \frac{\partial \Theta}{\partial x} - \Delta F,$$

in which

$$\Theta = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z},$$

$$\Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}.$$

Substituting this value in equation (13) we get

$$\mu 4\pi u' = \left(\frac{\partial \Theta}{\partial x} - \Delta F\right). \tag{14}$$

Eliminating u' between equations (12) and (14) and repeating the analogous operations for the other co-ordinates, we get finally

$$\begin{aligned}
&\mu\left(4\pi c + K\frac{\partial}{\partial t}\right)\left(\frac{\partial F}{\partial t} + \frac{\partial \psi}{\partial x}\right) + \frac{\partial \Theta}{\partial x} - \Delta F = 0, \\
&\mu\left(4\pi c + K\frac{\partial}{\partial t}\right)\left(\frac{\partial G}{\partial t} + \frac{\partial \psi}{\partial y}\right) + \frac{\partial \Theta}{\partial y} - \Delta G = 0, \\
&\mu\left(4\pi c + K\frac{\partial}{\partial t}\right)\left(\frac{\partial H}{\partial t} + \frac{\partial \psi}{\partial z}\right) + \frac{\partial \Theta}{\partial z} - \Delta H = 0.
\end{aligned} \tag{15}$$

Taking the partial differentials of these equations in reference to x, y, and z respectively, and adding them, we get

$$\mu\left(4\pi c + K\frac{\partial}{\partial t}\right)\left(\frac{\partial \Theta}{\partial t} + \Delta\psi\right) = 0. \tag{16}$$

When the medium is not a conductor, $c=0$, and the value of $\Delta\psi$, which is proportional to the density of the free electricity, is independent of the time. There remains then $\frac{\partial^2\Theta}{\partial t^2}=0$; that is to say that Θ is a linear function of the time, or a constant. These two functions Θ and ψ play no part therefore in the phenomena due to periodical disturbances.

629. PROPAGATION OF UNDULATIONS IN A DIELECTRIC.—In the case of a dielectric, equations (15) may be reduced to

$$\begin{aligned} K\mu\frac{\partial^2 F}{\partial t^2}-\Delta F&=0.\\ K\mu\frac{\partial^2 G}{\partial t^2}-\Delta G&=0,\\ K\mu\frac{\partial^2 H}{\partial t^2}-\Delta H&=0. \end{aligned} \tag{17}$$

These equations define the manner in which the functions F, G, and H vary with the time, and therefore the propagation of electromagnetic disturbances; they are of the same form as that of vibratory motion in a solid elastic body.

The velocity V of the propagation of a disturbance is given by the expression

$$V=\frac{1}{\sqrt{K\mu}}. \tag{18}$$

630. PLANE WAVES.—Suppose, in fact, that at an instant the electromagnetic disturbances form a plane wave perpendicular to the axis of z. The medium will be traversed by plane waves parallel to the first, and all the quantities, whose variations determine these waves, are simply functions of z and of t, independent of x and y. Equations (2) of (567) become then

$$\begin{aligned} \mu X&=\frac{\partial G}{\partial z},\\ \mu Y&=-\frac{\partial F}{\partial z},\\ \mu Z&=0. \end{aligned}$$

In like manner, the equations analogous to (13) give,

$$4\pi u' = -\frac{\partial Y}{\partial z} = -\frac{1}{\mu}\frac{\partial^2 F}{\partial z^2},$$

$$4\pi v' = -\frac{\partial X}{\partial z} = -\frac{1}{\mu}\frac{\partial^2 G}{\partial z^2}, \tag{19}$$

$$4\pi w' = 0.$$

We see already that the electrical perturbation is also in the plane of the wave, and perpendicular to the electromagnetic disturbance, for if we have $Y = 0$ (that is if the electromagnetic disturbance is parallel to the x axis), we shall have $u' = 0$, and the electrical disturbance will be parallel to the y axis.

For a non-conducting medium, equation (12) and the analogous relations to the other co-ordinates give

$$4\pi u' = -K\frac{\partial^2 F}{\partial t^2},$$

$$4\pi v' = -K\frac{\partial^2 G}{\partial t^2}, \tag{20}$$

$$4\pi w' = -K\frac{\partial^2 H}{\partial t^2};$$

from this follows

$$\frac{\partial^2 F}{\partial t^2} = \frac{1}{K\mu}\frac{\partial^2 F}{\partial z^2}, \quad \text{or} \quad \frac{\partial^2 F}{\partial t^2} = V^2\frac{\partial^2 F}{\partial z^2},$$

$$\frac{\partial^2 G}{\partial t^2} = \frac{1}{K\mu}\frac{\partial^2 G}{\partial z^2}, \qquad \frac{\partial^2 G}{\partial t^2} = V^2\frac{\partial^2 G}{\partial z^2}, \tag{21}$$

$$\frac{\partial^2 H}{\partial t^2} = 0; \qquad \frac{\partial^2 H}{\partial t^2} = 0.$$

The integral of these latter equations is

$$H = A + Bt, \tag{22}$$

an expression in which A and B are functions of z. This quantity H is therefore constant, or varies proportionally with the time. In any case it does not intervene in the propagation of periodical phenomena.

The integrals of the two first equations are expressions of the form

$$(23) \qquad \begin{aligned} F &= f_1(z - Vt) + f_2(z + Vt), \\ G &= \phi_1(z - Vt) + \phi_2(z + Vt). \end{aligned}$$

The values of F and of G consist of two distinct parts. The former does not change when we make successively $z = 0$, $t = 0$, or $z = V$, and $t = 1$; it represents a plane wave which moves parallel to the z axis with a velocity equal to V. The second also represents a plane wave which moves in the opposite direction with the same velocity.

631. A magnetic disturbance in the form of a plane wave produces then two plane waves moving on each side with the same velocity.

When $G = 0$, the magnetic force is parallel to the y axis and equal to $-\frac{\partial F}{\partial z}$; the electromotive force is parallel to the x axis and its value is $-\frac{\partial F}{\partial t}$. If we assume that the phenomena are identical with those of light, the present case corresponds to a ray of polarized light. The plane of polarization would coincide, either with the plane of magnetic disturbance, or with the plane of electrical disturbance which is perpendicular to it.

If the original disturbance is periodic, and forms a simple vibration proportional to $\sin 2\pi \frac{t}{T}$, the same character will be met with in each of the planes parallel to the original wave, and the wave-length of the phenomenon is the distance VT traversed by the propagation of the motion during a single period.

If the disturbance is circular, that is, if it may be figured by a movable body which describes a circumference in a uniform motion, the same character will be reproduced in the waves propagated, and the planes passing through the radius and the magnetic force, or the electrical displacement, are always perpendicular to each other. This would be the case with a circularly polarized ray of light.

632. DISTRIBUTION OF THE ENERGIES.—We have seen (120) that in an electrified system, the energy of the medium for unit volume is equal to half the product of the displacement by the electrical force. In like manner, in the field of a system of currents (570) the energy for unit volume is equal to the quotient of the square of the induction by $8\pi\mu$.

Suppose that the plane wave in question is polarized, and that the electrical disturbance is directed along the x axis. We shall then have $X=Z=0$; $v'=w'=0$; $g=h=0$; $Q=R=0$; $G=H=0$.

The electrical energy for unit volume is expressed by

$$\frac{1}{2}fP=\frac{K}{8\pi}P^2=\frac{K}{8\pi}\left(\frac{\partial F}{\partial t}\right)^2,$$

and the electromagnetic energy

$$\frac{1}{8\pi\mu}(\mu^2Y^2)=\frac{1}{8\pi\mu}\left(\frac{\partial F}{\partial z}\right)^2.$$

These two expressions are equal, for if we multiply the two members of the first of the equations (21) by the equal factors $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial z}\frac{\partial z}{\partial t}$, and integrate with respect to t, we get

$$\left(\frac{\partial F}{\partial z}\right)^2=K\mu\left(\frac{\partial F}{\partial t}\right)^2.$$

The total energy of the medium in which the waves are propagated is therefore half in the form of electrostatic energy, and half in the form of electromagnetic energy.

Let p denote each of these energies for unit volume. In virtue of its electrical state (104) the medium is subject to a tension $\frac{p}{2}$ parallel to the x axis, and a pressure of the same value parallel to the axis of x and z. In virtue of its electromagnetic condition, the medium is subject to the same actions, except that the x axis must be replaced by that of y, and conversely. These actions destroy themselves in the plane of the wave, and there remains a pressure p parallel to this plane equal to half the total energy for unit volume.

A ray of light produces therefore in the medium a pressure parallel to the direction of the motion, and would exert a repulsion on a plate of metal which it encountered. It is possible that this effect may have some part in the motion of the radiometer.

633. Velocity of the Propagation of Light.—The true control of this theory is, then, that in all media the velocity of

the propagation of the magnetic disturbances is the same as the velocity of light.

Let us suppose, in the first case, that the medium in question is air. The coefficient K would be equal to unity if the electrostatic units had been adopted. In the electromagnetic system, the value of this coefficient (608) is $\frac{1}{a^2}$. It follows from this that

$$V = \frac{1}{\sqrt{K}} = a.$$

Hence the velocity of the propagation of an electromagnetic disturbance in air is equal to the ratio of the units; this ratio ought then to be equal to the velocity of the propagation of light. Now experiment gives values for these two velocities which differ extremely little from 300,000 kilometres per second, and the most recent researches agree in giving numbers which are the nearer each other, the more exact the measurements have been. Such a coincidence cannot be due to accident, and Maxwell's theory finds thus a most striking experimental confirmation.

634. Specific Inductive Capacity.—Let us now consider a dielectrical medium, the refractive index of which is n, and its specific inductive capacity greater than that of air. If V is the velocity of the propagation of light in air, and V′ its velocity in the medium in question, we have

$$nV' = V, \qquad \text{or} \qquad n^2V'^2 = V^2.$$

On the other hand the velocity V″ of electromagnetic disturbance will be obtained if we replace K by K′, which gives

$$K'V''^2 = KV^2 = 1.$$

In order that V″ and V′ shall be equal, we must have $n^2 = \frac{K'}{K}$.

It follows then from this theory that the specific inductive capacity of a dielectric with respect to air is equal to the square of its refractive index.

A difficulty here presents itself in the experimental verification of this conclusion, which arises from the dispersion of refracting media. As the refractive index varies with the wave-length the most natural idea would be to take the limiting value of the index:

that is to say, that which corresponds to the greatest wave length. For paraffine, for instance, the refractive indices of the extreme luminous rays vary from 1.43 to 1.45, and the best experiments show that the specific inductive power is equal to 2.29, the square root of which 1.51 is not greatly different from the refractive index. The agreement is much less satisfactory with most of the transparent solid dielectrics, such as the different kinds of glass, Iceland spar, fluor spar, and quartz; their specific inductive capacity is always higher, and is sometimes twice the square of the refractive index. This is also the case with the animal and vegetable oils, according to the recent experiments of Dr. Hopkinson.

For gases, in which the refraction is less, and the dispersion may be neglected, the refractive power $n^2 - 1$ is proportional to the specific gravity, or to the pressure, if the temperature is constant; it ought to follow from this that the specific inductive capacity also increases in proportion to the pressure, and by the same coefficient as the refractive power. This conclusion seems to have been verified by the researches of M. Boltzmann.

The experimental control cannot therefore be considered sufficient to confirm the theory; but too much importance must not be attached to this apparent disagreement, if we take into account the fact that the specific inductive capacity diminishes with the duration of the electrification. But the period of electrical oscillations, which must be assumed in order to explain luminous phenomena, is out of all proportion with the shortest interval of time that can be realised in electrical experiments.

In any case, this correlation between the electrical and optical properties of a medium may be considered as, at all events, a first approximation to a theory which remains to be more minutely developed.

635. Anisotropic Media.—In order to extend the theory to anisotropic media, we should, in strictness, know the relation between the molecular constitution of a medium and its electrical properties; but without making any hypothetic supposition, it is sufficient if we assume that the specific inductive capacity of the medium is not the same in different directions; in other words, that the electromotive force, instead of being proportional to the displacement, and in the same direction, is connected with the displacement by a system of linear equations, as for the phenomena of thermal expansion.

In this case, there are three rectangular directions, along which the electromotive force is in the direction of the displacement;

if we take these directions as axes of co-ordinates, and call K_1, K_2, and K_3, the three principal values of the specific inductive capacity, or a, b, and c, the three principal velocities of propagation, we may write

$$4\pi f = K_1 P = \frac{1}{a^2} P,$$

$$4\pi g = K_2 Q = \frac{1}{b^2} Q, \tag{24}$$

$$4\pi h = K_3 R = \frac{1}{c^2} R.$$

In a non-conducting medium, where the electrical density is constant at each point, the general equations of propagation become then

$$K_1 \frac{\partial^2 F}{\partial t^2} = \Delta F - \frac{\partial \Theta}{\partial x} = \frac{1}{a^2} \frac{\partial^2 F}{\partial t^2},$$

$$K_2 \frac{\partial^2 G}{\partial t^2} = \Delta G - \frac{\partial \Theta}{\partial y} = \frac{1}{b^2} \frac{\partial^2 G}{\partial t^2}, \tag{25}$$

$$K_3 \frac{\partial^2 H}{\partial t^2} = \Delta H - \frac{\partial \Theta}{\partial z} = \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2}.$$

If l, m, and n are the cosines of the angles which the perpendicular to a plane wave, moving with the velocity V, makes with the axis, we may write

$$lx + my + nz - Vt = w.$$

If we represent by F″, G″, H″ the second differentials in respect of w of the different functions F, G, and H; equations (25) become

$$\left(\frac{V^2 - a^2}{a^2} + l^2\right) F'' + l\,(mG'' + nH'') = 0,$$

$$\left(\frac{V^2 - b^2}{b^2} + m^2\right) G'' + m(nH'' + lF'') = 0, \tag{26}$$

$$\left(\frac{V^2 - c^2}{c^2} + n^2\right) H'' + n(lF'' + mG'') = 0.$$

Eliminating the functions F″, G″, and H″ between these three equations, we get

$$\frac{l^2}{V^2-a^2}+\frac{m^2}{V^2-b^2}+\frac{n^2}{V^2-c^2}=0. \tag{27}$$

This is an equation of the same form as that which determines the velocity of propagation of light in double refracting media. For a given direction it gives two velocities corresponding to two distinct waves which move in the same direction. If the wave, for instance, is perpendicular to the x axis, we have $m=0$, $n=0$, and the values of the velocity are b and c.

636. If the medium is symmetrical in reference to one axis (for instance, the x axis), the two velocities b and c are equal, and equation (27) reduces to

$$l^2(V^2-b^2)^2+(1-l^2)(V^2-a^2)(V^2-b^2)=0.$$

For a wave perpendicular to the axis, $l=1$; there is then only one velocity of propagation, $V=b$, whatever be the direction of the electrical and magnetic disturbances in the plane of the wave. If the wave is parallel to the axis, $l=0$, and we have two velocities of propagation, $V=a$, and $V=b$. The wave which is propagated with the velocity b, has the character of the ordinary wave in optical phenomena, and corresponds to an electrical disturbance perpendicular to the axis. As this wave is polarized in the plane of the axis and of the ray, we see that the plane of polarization of the light is perpendicular to the plane of the electrical disturbance.

637. Bad Conductors.—Let us suppose that the medium is an imperfect insulator, and is isotropic, and that the effects of displacement and of conductivity are of the same order. The energy is then partially transformed into heat, and the wave which is propagated is gradually weaker.

Let us consider a plane wave perpendicular to the z axis, the disturbance being parallel to the x axis. Equations (15) give then

$$4\pi c\frac{\partial F}{\partial t}+K\frac{\partial^2 F}{\partial t^2}=\frac{\partial^2 F}{\partial z^2}. \tag{28}$$

The integral is a function of the form

$$F=e^{-pz}\cos(nt-qz),$$

and the coefficients ought to satisfy the conditions

$$q^2 - p^2 = Kn^2,$$
$$2pq = 4\pi cn.$$

This expression represents a wave which is propagated parallel to the z axis, with a velocity V equal to $\frac{n}{q}$, and the amplitude of which rapidly diminishes.

The value of the coefficient of absorption p is $2\pi c$V. The absorption of light ought therefore to increase with the electrical conductivity. Experiment does, in fact, show that most transparent substances are dielectrics, and that all good conductors are very opaque.

This ratio, however, is not absolute, for certain metals are transparent in very slight thicknesses, and many dielectrics are opaque. We ought to exclude electrolytes, which are almost all transparent, for the decomposition which accompanies the passage of electricity completely changes the nature of the phenomenon, and we are no longer dealing with a mere effect of conductivity.

638. CONDUCTING BODIES.—Let us consider, finally, an isotropic conducting medium, or at any rate a medium in which the phenomena of conduction predominate over those of electrical displacement. If we neglect then the terms containing the factor K, in equations (15) as well as the functions Θ and ψ, we get

$$4\pi c \frac{\partial F}{\partial t} = \Delta F,$$
$$4\pi c \frac{\partial G}{\partial t} = \Delta G. \qquad (29)$$
$$4\pi c \frac{\partial H}{\partial t} = \Delta H.$$

Each of these equations has the same form as that which gives the diffusion of heat in Fourier's theory.

For, if k is the coefficient of thermal conductivity of an isotropic medium (70), and if the function F be considered as giving the temperature at each point, the expression $k\Delta F$ represents the flow of heat which in unit time penetrates unit volume; $\frac{\partial F}{\partial t}$ is the corresponding rise of temperature, so that the coefficient $4\pi ck$ is the calorific capacity for unit volume.

Electromagnetic properties, once established in a medium, experience therefore a diffusion analogous to that of heat; but we must remark that the coefficient of conductivity of the medium k, which would produce the same calorific diffusion, is inversely as c. The diffusion of the electromagnetic effects is then inversely as the electrical conductivity, so that a medium which had a perfect conductivity would offer an absolute obstacle to this diffusion.

Consider, for instance, the case of a linear conductor surrounded by a conducting medium. The moment the principal current is established, the induced current in the surrounding medium has the same strength, and their action on a distant point is null; the permanent state is only established after the induced currents have been nullified by the resistance of the medium. But in the degree in which the induced current is enfeebled, a current in the same direction is produced round it, so that the space occupied in the medium by the induced current increases in proportion as the intensity diminishes.

If the principal current is kept constant, the induced currents diffuse gradually; when the permanent regime is set up, the values of ΔF, ΔG, and ΔH are null throughout the medium, and only retain finite values in the portion occupied by the circuit of the current.

639. ROTATORY MAGNETIC POLARIZATION.—The ordinary theory of undulations assumes that luminous phenomena are produced by the vibrations of ether; but it must be admitted that a formal explanation of this kind is outside the range of experiments. We do not know, in fact, what is the true nature of light. The only thing which may be considered to be proved, is that in a ray of light there is a mechanical effect of the nature of a vector in geometry, that is characterized by a magnitude and a direction; this direction is perpendicular to the ray, and it varies periodically in the same plane when the ray is polarized. This is a conclusion from the phenomena of interference.

In the case of a circularly polarized ray, the magnitude of this mechanical effect, of this vector, is constant, but its direction turns about the ray, and produces a complete revolution in each period. When such a ray traverses a medium under the action of a magnetic force, its velocity of propagation is modified; it must be concluded thence that there is, in the medium, some rotatory motion, the axis of which is parallel to the direction of the magnetic forces. This rotation does not apply to any finite portion of the medium taken as a whole, and it must be assumed that it is confined to the

smallest particles of bodies, each of which turns about its own axis. This is the hypothesis (601) of molecular vortices.

Maxwell explained in this way the phenomena of rotatory polarization by the conception of molecular vortices; but, without entering upon an explanation of this theory, we may arrive at the same result, as Professor Rowland showed, by bringing in a new action discovered by Mr. Hall.

640. HALL'S PHENOMENON.—Let ABCD (Fig. 127) be a cross cut in a very thin metal sheet—a gold leaf, for instance; the two ends A and B of the principal branch are connected with the poles of a battery, the ends C and D of the cross piece are connected with a galvanometer. The apparatus may easily be arranged so that none of the current traverses the galvanometer.

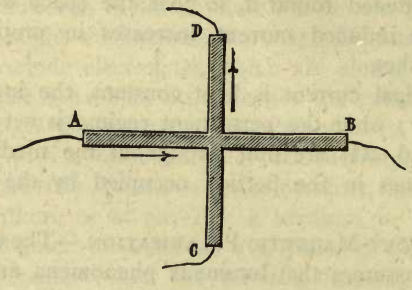

Fig. 127.

When this conductor is placed in a very strong magnetic field, so that the lines of force are perpendicular to its plane, a permanent deflection of the galvanometer shows that a constant current traverses the galvanometer. If the current goes from A to B in the principal branch, and the lines of force traverse the plane of the figure from front to back, the branch current goes through the galvanometer from D to C, when the conductor consists of gold, silver, platinum, or tinfoil, and in the opposite direction when the metal is iron. The action ceases to be perceived when the thickness of the conductor is increased.

In the first case, the current is drawn in the direction of the electromagnetic force which would be exerted on a wire parallel to AB, and traversed by a current from A to B; this may also be said to be the same in the second case, since in the interior of an iron plate, owing to the magnetisation, the direction of the lines of force, and of the electromagnetic force, have changed their sign.

Explained in this manner, Hall's phenomenon would seem to be in contradiction with the opinion generally adopted, that in electromagnetic phenomena the action is exerted on the *supports* of the currents, and not on the current itself. But, however we may explain the experiment, it follows that a magnetic field in the stationary state develops an electromotive force which tends to move electricity in the direction of the electromagnetic action—that is, to the left of an observer placed in the current, and who is looking in the direction of the magnetic force.

As the effect in question is very small, the most natural hypothesis, which moreover is approximately verified by Hall's experiments, is to assume that it is proportional to the electromagnetic force.

641. General Equations.—Let A, B, C be the components of this new electromotive force, and suppose that we are dealing with a magnetic medium. The component A, which acts along the x axis, is the algebraical sum of the two actions exerted on the components v' and w' of the flow of electricity along the y axis, and along the z axis; the former is proportional to $-Zv'$ and the second to Yw'. If γ is the coefficient of proportionality we shall have

$$\begin{aligned} A &= \gamma(Yw' - Zv'), \\ B &= \gamma(Zu' - Xw'), \\ C &= \gamma(Xv' - Yu'). \end{aligned} \tag{30}$$

The components of the total electromotive force of the field become then

$$\begin{aligned} P &= -\frac{\partial F}{\partial t} + A, \\ Q &= -\frac{\partial G}{\partial t} + B, \\ R &= -\frac{\partial H}{\partial t} + C. \end{aligned} \tag{31}$$

Introducing the new electromotive forces, equations (15) give

$$\mu\left(4\pi c + K\frac{\partial}{\partial t}\right)\left(\frac{\partial F}{\partial t} - A + \frac{\partial \psi}{\partial x}\right) + \frac{\partial \Theta}{\partial x} - \Delta F = 0, \text{ etc.} \tag{32}$$

As the functions ψ and Θ play no part in the periodical phenomena if the medium is not a conductor, we shall have

$$(33)\qquad \begin{aligned} K\mu\left(\frac{\partial^2 F}{\partial t^2}-\frac{\partial A}{\partial t}\right)&=\Delta F,\\ K\mu\left(\frac{\partial^2 G}{\partial t^2}-\frac{\partial B}{\partial t}\right)&=\Delta G,\\ K\mu\left(\frac{\partial^2 H}{\partial t^2}-\frac{\partial C}{\partial t}\right)&=\Delta H. \end{aligned}$$

642. PROPAGATION OF A PLANE WAVE.—Let us consider a plane wave perpendicular to the z axis. We need only take into account the component Z of the magnetic force parallel to this axis; in this case, if the field is constant, equations (33) reduce to

$$(34)\qquad \begin{aligned} K\mu\left(\frac{\partial^2 F}{\partial t^2}+\gamma Z\frac{\partial v'}{\partial t}\right)&=\frac{\partial^2 F}{\partial z^2},\\ K\mu\left(\frac{\partial^2 G}{\partial t^2}-\gamma Z\frac{\partial u'}{\partial t}\right)&=\frac{\partial^2 G}{\partial z^2},\\ K\mu\frac{\partial^2 H}{\partial t^2}&=0. \end{aligned}$$

Equations (2) and (12) of Arts. 567 and 572 give

$$\begin{aligned} 4\pi u'&=+\frac{\partial Y}{\partial z}=-\frac{1}{\mu}\frac{\partial^2 F}{\partial z^2},\\ 4\pi v'&=-\frac{\partial X}{\partial z}=-\frac{1}{\mu}\frac{\partial^2 G}{\partial z^2}. \end{aligned}$$

From this follows

$$\begin{aligned} \frac{\partial u'}{\partial t}&=-\frac{1}{4\pi\mu}\frac{\partial^3 F}{\partial z^2\partial t},\\ \frac{\partial v'}{\partial t}&=-\frac{1}{4\pi\mu}\frac{\partial^3 G}{\partial z^2\partial t}. \end{aligned}$$

Substituting these values in equations (34), we get

$$(35)\qquad \begin{aligned} K\mu\left(\frac{\partial^2 F}{\partial t^2}-\frac{\gamma Z}{4\pi\mu}\frac{\partial^3 G}{\partial z^2\partial t}\right)&=\frac{\partial^2 F}{\partial z^2},\\ K\mu\left(\frac{\partial^2 G}{\partial t^2}+\frac{\gamma Z}{4\pi\mu}\frac{\partial^3 F}{\partial z^2\partial t}\right)&=\frac{\partial^2 G}{\partial z^2}. \end{aligned}$$

These equations have a solution of the form

$$(36)\qquad \begin{aligned} F&=r\cos(pt-qz)\cos mz,\\ G&=r\cos(pt-qz)\sin mz. \end{aligned}$$

The coefficients will be determined by the condition that the differential equations are satisfied for all values of z and of t, which gives

$$(37)\qquad \begin{aligned} K\mu(p^2+m^2)-q^2\left(1+\gamma\frac{mZK}{4\pi}\right)&=0,\\ 2m\mu-\gamma\frac{Zq^2}{4\pi}&=0. \end{aligned}$$

The values of F and G are projections on the axes of an electromotive force $r\cos(pt-qz)$, which makes with the axis of x an angle mz proportional to the thickness.

643. ROTATION OF THE PLANE OF POLARIZATION. — The phenomenon represents then a ray of light, which is propagated along the z axis with a velocity $V=\frac{p}{q}$, the period of whose vibration is $T=\frac{2\pi}{p}$; the wave length—that is, the space traversed during a period—is $\lambda=\frac{p}{q}\frac{2\pi}{p}=\frac{2\pi}{q}$.

This ray is rectilinearly polarized; but the plane of polarization rotates as in an active medium, and the complete rotation is effected in a time $\tau=\frac{2\pi}{m}$.

The equations (37) give

$$m = \frac{\gamma}{2\mu}\frac{Zq^2}{4\pi} = \frac{\pi\gamma Z}{2\mu\lambda^2},$$

$$V = \frac{p}{q} = \frac{1}{\sqrt{K\mu}}\sqrt{1+\gamma\frac{mZK}{8\pi}} = \frac{1}{\sqrt{K\mu}}\sqrt{1+\frac{K\mu m^2\lambda^2}{4\pi^2}}.$$

As the term in γ is very small, we may take the square root as an approximation, and we get finally:

$$m = \gamma\frac{\pi Z}{2\mu\lambda^2},$$

$$V = \frac{1}{\sqrt{K\mu}}\left(1+\frac{K\mu m^2\lambda^2}{8\pi^2}\right),$$

$$\tau = \frac{4\mu\lambda^2}{\gamma Z}.$$

We draw from this the following conclusions:—

1st. When a polarized ray travels along the direction of a magnetic line of force, the plane of polarization is rotated in a direction which depends on the sign of γ.

This is the phenomenon discovered by Faraday, with the inversion for magnetic bodies observed by Verdet.

2nd. The velocity of propagation is increased by the electromagnetic action; but this effect is no doubt too feeble to be made evident.

If λ_0 and V_0 are the wave length and the velocity of the ray in vacuum, when it is withdrawn from the magnetic action, and n the refractive index on passing into the medium in question, we have $n\lambda = \lambda_0$ and $nV = V_0$. The time which this ray takes in traversing a thickness e of the medium is $\frac{e}{V} = \frac{ne}{V_0}$; the rotation θ which the plane of polarization experiences is

$$\theta = m\frac{e}{V} = \gamma eZ\frac{\pi n^3}{2\mu V\lambda^2},$$

and the rotation ω, for unit difference of potential, becomes

$$\omega = \gamma \frac{\pi n^3}{2\mu V \lambda^2}.$$

If the refractive index were independent of the dispersion of the medium, the rotation of the plane of polarization will be inversely as the square of the wave length, which is approximately the law of magnetic rotation.

We have seen (597) that, to allow for the dispersion of the medium, this result must be multiplied by the factor $\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$; we get then

$$\omega = \gamma \frac{\pi n^2}{2\mu V \lambda^2}\left(n - \lambda \frac{dn}{d\lambda}\right).$$

This, apart from the factor μ, is the formula at which Maxwell had arrived by the theory of molecular vortices, and that which best accords with experiment. We see that, other things being equal, the rotatory power is inversely as the coefficient of permeability μ.

CHAPTER XI.

SUPPLEMENTARY.

644. CONCLUSIONS FROM CARNOT'S PRINCIPLE. — Sir W. Thomson showed that the principles which serve as the basis of the theory of heat—that is to say, the principle of the conservation of energy and Carnot's principle—render it possible to establish some important properties relating to electrical and magnetic phenomena.

Whenever a body loses heat or changes its dimensions, in opposition to external forces which tend to deform it in the contrary directions, it does work. Whatever be the cycle of transformation, the external mechanical work only depends on the initial and final state of the body. In all cases, this mechanical or calorific work corresponds to a loss of energy of the body in question.

The intrinsic or potential energy of a body is the total work which it could do if it were indefinitely cooled, or if it were expanded or contracted to an unlimited extent, according as the molecular forces are attractive or repulsive. There is no means of measuring this energy, nor even of knowing whether it has a finite value for a limited mass; but we may measure the changes which it undergoes, starting from a determinate condition, which is taken as the *normal state.*

The mechanical state of a homogeneous body which has undergone any homogeneous deformation — that is to say, a deformation which is reproduced in the same manner in each element of volume — may be expressed by six independent variables: for instance, the lengths of the sides and the value of the angles of a parallelopipedon, or the six elements of an ellipsoid, which would always contain the same portion of a solid.

645. The potential energy E of a body for unit weight, starting from the normal state, is a function of its mechanical condition

(form and dimensions) and of its temperature. When the body undergoes any transformation, it absorbs a certain quantity H of energy which depends on the deformation it has experienced, and on the variation of temperature. If only one of the variables x, varies by dx, and the temperature by $d\mathrm{T}$, we may write

$$d\mathrm{H} = a\,dx + l\,d\mathrm{T}. \tag{1}$$

The functions a and l have an obvious physical meaning. If we divide them by the mechanical equivalent of heat, the former represents the latent heat relative to the variable x, and the second l the specific heat for a constant mechanical state.

The work $d\mathrm{W}$ done by external forces only depends on the change of form, and we have

$$d\mathrm{W} = b\,dx.$$

The increase of potential energy is the sum of these two expressions, which gives

$$d\mathrm{E} = (a+b)\,dx + l\,d\mathrm{T}. \tag{2}$$

For any given closed cycle, the total variation of potential energy is null; the elementary variation must then be an exact differential of the independent variables, which gives the condition

$$\frac{\partial(a+b)}{\partial \mathrm{T}} = \frac{\partial l}{\partial x}. \tag{3}$$

This equation may be considered as expressing the principle of the conservation of energy, or the mechanical equivalence of heat.

646. In order to apply Carnot's principle, the cycle of transformations must be reversible, and the final state of the body must be identical with the initial state.

The sum of the quotients of the calorific energy absorbed by the corresponding absolute temperature is then null, and we have

$$\int \frac{d\mathrm{H}}{\mathrm{T}} = \int \left(\frac{a}{\mathrm{T}}\,dx + \frac{l}{\mathrm{T}}\,d\mathrm{T} \right) = 0.$$

The expression in the parenthesis should also be an exact differential, from which it follows that

$$\frac{\partial a}{\partial T} - \frac{a}{T} = \frac{\partial l}{\partial x}. \tag{4}$$

This equation (4) may also be considered as the translation of Carnot's principle.

Comparing equations (3) and (4), we deduce

$$\frac{a}{T} = -\frac{\partial b}{\partial T}. \tag{5}$$

An analogous equation would be obtained for any other independent variable. If A is the whole differential coefficient of the calorific energy absorbed for any transformation of the body at a constant temperature, which corresponds to the latent heat relative to this transformation, and if B is the corresponding differential coefficient of the external work, we shall have

$$A = -T\frac{\partial B}{\partial T}. \tag{6}$$

647. We may deduce the following conclusions from this equation (6):—

Whenever the external work takes place in a direction such that the rate of change $\frac{\partial B}{\partial T}$ is negative, the value of A is positive. In other words, whenever the deformation produced by the external work is such that a deformation of the same kind could be produced by a cooling of the body, this work is accompanied by a disengagement of heat, and therefore by a rise of temperature. The reverse is the case if the rate of change $\frac{\partial B}{\partial T}$ were positive. Hence it is that a gas becomes heated when it is compressed, and cooled when it expands.

As solids expand as a general rule when the temperature rises, we see that the uniform compression of a solid will also produce a rise of temperature.

It is otherwise with bodies whose expansion is abnormal, such as water at a temperature below 4°, and iodide of silver at ordinary temperatures. The compression of these bodies would produce a lowering of temperature.

In like manner also a stretched metal wire should become cooled when it is twisted to a greater extent, if we assume as certain that the coefficient of torsion diminishes as the temperature rises. A twisted wire ought also to become more heated, independently of the external work, when it is allowed to untwist.

In any case, the amount of energy H, absorbed or disengaged, may be deduced from the external work and the properties of a body. Without dwelling further on the purely calorific phenomena which could thus be deduced from Carnot's principle, we shall examine some deductions from equation (6) relative to electrical or magnetic phenomena.

648. CHANGES OF TEMPERATURE DURING MAGNETISATION.—The relations we have already pointed out between the changes of temperature and the coefficients of magnetisation enable us to predict the following results:—

1st. If we work at a temperature below redness, but so high that the coefficient of magnetisation is decreasing, a piece of soft iron should become heated when it is slowly brought near a magnet, and cooled when it is removed. We assume that the motion is slow, so as to avoid the influence of induction currents.

The reverse would be the case at ordinary temperatures, if the coefficient of magnetisation, as seems probable, increases with the temperature.

2nd. Cobalt should behave like iron—that is, become cooled when it is brought near a magnet at the ordinary temperature; and become heated, on the contrary, if we work at a higher temperature than that of the maximum of magnetisation.

3rd. For nickel, there is no maximum of magnetisation; at all temperatures this metal ought to become heated when it is brought near, and cooled when it is moved away from a magnet.

More generally, nickel and cobalt at ordinary temperatures ought to become cooled when the motion requires an external work opposed to that of the magnetic forces. For nickel at any given temperature, and for the two former metals, at temperatures higher than those of the maximum of magnetisation, any displacement which requires a work opposed to magnetic actions produces, on the contrary, a heating of the body.

4th. In a magnetic field, a crystal becomes cooled when its axis of greatest magnetic induction, or of least diamagnetic induction, passes from a direction parallel, to a direction perpendicular to that of the field.

649. Electrical Heating of Tourmaline.—Pyroelectrical phenomena give rise to analogous considerations.

The pyroelectricity of crystals is explained, on Faraday's theory (118), by assuming that the crystal is in a state of electrical polarization, the external effect of which is equivalent to that of two layers, of equal masses and contrary signs, distributed on the surface. When the crystal is at a constant temperature, the surrounding medium, either by its own conductivity, or by the surface of the crystal itself, soon acquires a superficial electrification, which neutralises the former, and annuls its action on any external point.

When the crystal is broken perpendicularly to the electrical axis, the whole of each of the two fragments then are electrified in opposite directions, not only by the new layers which the polarization on the broken surfaces produces, but also in consequence of the electrification induced on the old surfaces, the equilibrium of which is broken.

When the temperature changes, the electrification changes also; but the equilibrium produced by the electrification of the surrounding medium is only set up by degrees, and more or less slowly, according to the conductivity of the medium or of the surface of the crystal.

If this explanation of pyroelectricity is correct, it follows that a pyroelectrical crystal should be heated or cooled when it is moved in an electrical field, like magnetised iron in a magnetic field.

Tourmaline becomes heated if it is displaced in such a manner that the influence of the field tends to increase its polarization, and is cooled in the contrary case.

The effect produced on tourmaline does not depend on the electrification of the surface, and we arrive at this remarkable result; a pyroelectrical crystal which appears in the natural state, its properties being neutralised by the electrification of the medium, undergoes, when it is displaced in a field, the same variations of temperature as if its electrical properties were apparent—that is to say, as if it had been raised to a high temperature, then dried, and rapidly cooled.

650. Principle of the Conservation of Electricity.—Whenever a system of bodies, disconnected from any external bodies, is the seat of any electrical phenomenon (8), the total quantity of electricity which it contains remains unchanged. This principle is verified in all experiments, and it is a consequence of Maxwell's views as to the constitution of the medium which serves for conveying electrical force. Although we are not able to affirm that

the total quantity of electricity in nature is strictly null, we may assume at least that actual physical phenomena produce no change, and that it remains constant in the same sense as the total quantity of energy or of matter. In other words, a quantity of electricity may be considered indestructible by any other cause than a quantity of electricity of the opposite sign. M. Lippmann has showed that this principle leads to conclusions analogous to those of Carnot's theorem ; when it is associated with the principle of the conservation of energy, we may deduce the explanation of a certain number of phenomena, and moreover predict other phenomena which have not yet been observed.

Suppose that a body A traverses a closed cycle in a system—that is to say, that, after having undergone a series of transformations, it returns to its primitive state—the sum of the quantities dm of electricity which it has received during the cycle is zero, so that we have

$$\int dm = 0. \tag{7}$$

This condition necessitates that the elementary increment dm can be integrated—that is to say, is the exact differential of a function of the independent variables. If the phenomenon only depends on two variables x and y, which is the most general case, we may write

$$dm = \mathrm{X}dx + \mathrm{Y}dy, \tag{8}$$

and the condition of integrability is

$$\frac{\partial \mathrm{X}}{\partial y} = \frac{\partial \mathrm{Y}}{\partial x}. \tag{9}$$

Equation (9) may be considered as in some sense the expression of the principle of the conservation of electricity; following M. Lippmann, we shall give some of the applications.

651. Electrocapillary Phenomena.—Lippmann has observed that the capillary effects manifested between mercury and acidulated water depend on the difference of potential of the two liquids, and conversely that the difference of potential of the liquids is modified when the magnitude of the surface of contact is changed by external forces. This reciprocity of phenomena is a consequence of the preceding principle.

Let S be the surface of mercury in contact with acidulated water, A the capillary tension of the liquid, and x the excess of potential of the water over that of the mercury; when, for any cause whatever, the surface is increased by dS, the work of the capillary tension is $-\mathrm{A}d\mathrm{S}$.

Let us now suppose that a quantity of electricity dm, furnished by an extraneous source, reaches the surface of the water; there will be an increase dx of the difference of potential, at the same time as a dilatation dS of the surface. The quantities x and S are then independent variables of the phenomenon. Since, other things being equal, the mass dm is proportional to the surface, we may write

$$dm = \mathrm{Y}\mathrm{S}dx + \mathrm{X}d\mathrm{S}. \tag{10}$$

The factor X represents the capacity of unit surface at constant potential, and Y the electrical capacity of unit surface, the surface being constant and the potential variable. The principle of the conservation of electricity gives the condition

$$\mathrm{Y} = \frac{\partial \mathrm{X}}{\partial x}. \tag{11}$$

On the other hand, the electrical work xdm, introduced into the system, produces an increase of potential energy of the surface, and an external work $-\mathrm{A}d\mathrm{S}$. If this operation be repeated several times in opposite directions, so as to return to the initial state, and that there has been neither gain nor loss of heat, the variation of energy of the system will be null, which gives

$$\int (x dm + \mathrm{A}d\mathrm{S}) = 0,$$

that is to say, replacing dm by its value,

$$\int \left[x\mathrm{Y}\mathrm{S}dx + (\mathrm{A} + x\mathrm{X})d\mathrm{S}\right] = 0. \tag{12}$$

As this expression must be null for any closed circuit, we have also

$$x\mathrm{Y} = \frac{\partial(\mathrm{A} + x\mathrm{X})}{\partial x}. \tag{13}$$

From these two equations (11) and (13) we deduce

$$X = -\frac{\partial A}{\partial x}, \qquad Y = -\frac{\partial^2 A}{\partial x^2};$$

and therefore

$$dm = -S\frac{\partial^2 A}{\partial x^2}dx - \frac{\partial A}{\partial x}dS = -d\left(S\frac{\partial A}{\partial x}\right).$$

The capacities X and Y are therefore functions of the capillary tension. It follows from this that if this tension is a function of the difference of potential, as experiment shows, this capacity cannot be null. If the surface is deformed, while the difference of potentials is kept constant, there should from equation (10) be a production or absorption of electricity; and, if we work at a constant charge, we modify the difference of potentials. These two orders of phenomena are therefore correlated, which is what experiment confirms.

652. GAS CONDENSERS.—Boltzmann has proved, in conformity with Maxwell's theory (634), that the capacity of a condenser, whose two coatings are separated by a layer of gas, varies proportionally with the pressure. The converse must follow, that the pressure of a definite mass of gas, placed between the coatings of a condenser, is a function of the difference of potential.

The two independent variables on which the phenomenon here depends, are the difference of potential x, and the pressure p of the gas. When the positive armature receives a quantity dm of electricity, we have

$$dm = Cdx + hdp. \tag{14}$$

The factor C represents the capacity of the condenser at constant pressure, h a coefficient which experiment shows is positive, for the capacity increases with the pressure, and which is determined by Maxwell's theory. The principle of the conservation of electricity gives

$$\frac{\partial C}{\partial p} = \frac{\partial h}{\partial x}. \tag{15}$$

Let us now consider a closed cycle, without change of temperature. The work required to increase by dm the charge of the positive coating is equal to xdm; on the other hand, a mass of

gas in contact with the condenser produces an external work pdv, when its volume increases by dv. If the gas and the condenser return to their original condition, the change of energy of the system is null, and we have

$$\int (x dm - p dv) = 0, \tag{16}$$

which requires that the expression within the parenthesis shall be an exact differential.

The volume v of the gas in question is a function of the pressure p, and perhaps also of the difference of potential x; we shall therefore write

$$dv = a dx + b dp, \tag{17}$$

and the course of the reasoning will show whether the coefficient a differs from zero. As this expression is also an exact differential, we deduce from it

$$\frac{\partial a}{\partial p} = \frac{\partial b}{\partial x}. \tag{18}$$

Substituting in the expression (16) the values of dv and of dm, we get

$$\int \left[(Cx - ap)\, dx + (hx - bp)\, dp \right] = 0.$$

The condition of integrability is then

$$\frac{\partial (Cx - ap)}{\partial p} = \frac{\partial (hx - bp)}{\partial x}. \tag{19}$$

Taking into account equations (15) and (18), this condition reduces to

$$a = -h.$$

The coefficient a is thus different from zero, and negative. It follows then, from equation (17), that, at a constant pressure, the volume of a mass of gas surrounding a condenser should diminish proportionally to the difference of potential of the armatures. This result seems to have been verified by Quincke—at any rate for carbonic acid.

653. ELECTRICAL DILATATION OF GLASS.—A Leyden jar dilates when it is electrified, and contracts as soon as it is discharged. This phenomenon, foreseen by Volta, was demonstrated by M. Govi, and M. Duter has shown that the expansion of gas is proportional to the square of the difference of potential of the armatures. Let us consider the phenomenon in the form given to it by M. Righi, that is to say, a condenser formed of a glass tube whose two faces are covered with tinfoil; let l be its length, and let us suppose that at the same time the tube is exposed to a strain in the direction of its length, which is represented by the weight p.

As the length is a function of the difference of potential x, and of the stretching weight p, we have

$$dl = a\,dx + b\,dp. \tag{20}$$

The coefficient a is positive, and measures the electrical elongation, and b is the coefficient of elasticity of the tube.

If we assume that the tube undergoes no permanent deformation, it follows that

$$\frac{\partial a}{\partial p} = \frac{\partial b}{\partial x}. \tag{21}$$

On the other hand, it is to be presumed that the charge of the condenser is also a function of the stretching weight, so that we may put

$$dm = \mathrm{C}\,dx + h\,dp, \tag{22}$$

C being the capacity of the jar, and h a coefficient which we do not know *à priori* to be different from zero.

The principle of the conservation of electricity gives

$$\frac{\partial \mathrm{C}}{\partial p} = \frac{\partial h}{\partial x}. \tag{23}$$

The variation of energy of the jar, for an increase dm of the charge, and an elongation dl, is

$$x\,dm + p\,dl = (\mathrm{C}x + ap)\,dx + (hx + bp)\,dp.$$

As this expression must be an exact differential, we deduce from it

$$\frac{\partial(Cx+ap)}{\partial p}=\frac{\partial(hx+bp)}{\partial x}, \tag{24}$$

or, taking equations (21) and (23) into account,

$$h=a.$$

It follows from equation (22), that at a constant potential, the charge of electricity increases with the stretching weight, and that with a constant charge, the potential diminishes when the stretching weight increases—that is to say, when the tube is elongated.

Experiment further shows that the elongation is proportional to the square of the difference of potential, and that, k being a constant,

$$\Delta l=kx^2.$$

It follows from this, that

$$a=\frac{\partial l}{\partial x}=2kx=h,$$

and, from equation (23),

$$C=C_0+2kp.$$

The capacity of the jar increases, therefore, proportionally to the stretching weight.

It may be observed that the electrical attraction of the two armatures of a condenser would also crush the intermediate layer, and give analogous effects; but it does not seem that this cause is sufficient to explain the phenomena.

654. Compression of Tourmaline.—We shall analyze in the same way a phenomenon recently discovered by MM. P. and J. Curie. When a tourmaline is compressed along its axis, the crystal becomes electrically polarized in the same direction as that which would be produced by a rise in temperature. This polarization is proportional to the compression, and disappears with it. Other hemihedral crystals, such as quartz and topaz, behave like tourmaline when they are compressed along a hemihedral axis.

Let us suppose that the bases of a prism of tourmaline are covered with plates of metal A and B, one of which B is connected with the earth, while the other can be connected with a source of constant potential.

The pressure p and the potential x of the armature A, may thus be varied by making the system pass through a closed cycle, and we may take these two quantities as independent variables. The quantity dm of electricity which the plate A receives is expressed by

$$dm = C dx + h dp.$$

The coefficient C is the capacity of the armature A at constant pressure, and h is a negative coefficient if the end A of the crystal is positively electrified by compression. The principle of the conservation of electricity gives then

$$\frac{\partial l}{\partial p} = \frac{\partial h}{\partial x}. \tag{25}$$

If l is the length of the crystal, we may also form the equation

$$dl = a dx + b dp,$$

in which b is the coefficient of elasticity of the crystal.

If we apply the principle of the conservation of energy as we have done above for M. Righi's tube, we deduce from it

$$a = -h.$$

As the value of h is negative, equation (26) shows that a tourmaline becomes longer, when it is electrified in the same way as it would be by a rise of temperature, and this elongation is proportional to the potential. A change of structure results, and, no doubt, also an alteration of optical properties analogous to that observed by Dr. Kerr in transparent bodies.

The factor h is constant, since, according to MM. Curie, the electrification is proportional to the pressure; hence the capacity of a condenser with a plate of tourmaline is independent of the compression to which the crystal is subjected.

655. Generalisation of Lenz's Law.—In all the preceding cases the converse of the phenomenon, the existence of which is

demonstrated by the principle of the conservation of electricity, is of such a nature as to oppose the production of the original phenomenon. We thus rediscover, in a more general way, Lenz's law (511) relative to the phenomena of induction.

These few examples will be sufficient to show how the general principles of the science enable us to connect the most varied phenomena, and even to determine their numerical relations, without its being necessary to know the intimate nature of the forces which come into play.

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